MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 9

1 (SHE 5.2)

4. The function $f(x) = 1 - x^2$ is decreasing on the interval $[0,1]$, so $m_i = f(x_i)$ and $M_i = f(x_{i-1})$ for all i. Hence,

$$L_f(P) = \tfrac{1}{4}\cdot\tfrac{15}{16} + \tfrac{1}{4}\cdot\tfrac{3}{4} + \tfrac{1}{2}\cdot 0 = \tfrac{27}{64}, \qquad U_f(P) = \tfrac{1}{4}\cdot 1 + \tfrac{1}{4}\cdot\tfrac{15}{16} + \tfrac{1}{2}\cdot\tfrac{3}{4} = \tfrac{55}{64}.$$

10. The function $f(x) = \cos x$ is decreasing on the interval $[0,\pi]$, so $m_i = f(x_i)$ and $M_i = f(x_{i-1})$ for all i. In similar fashion to the solutions above, we get

$$L_f(P) = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + 0\cdot\tfrac{\pi}{6} + (-1)\cdot\tfrac{\pi}{2} = -\tfrac{\pi}{3}, \qquad U_f(P) = 1\cdot\tfrac{\pi}{3} + \tfrac{1}{2}\cdot\tfrac{\pi}{6} + 0\cdot\tfrac{\pi}{2} = \tfrac{5\pi}{12}.$$

32. Suppose $P = \{x_0, x_1, \cdots, x_n\}$ is a regular partition of $[a,b]$. Then $x_k - x_{k-1} = \Delta x$ for all $k = 1, \cdots, n$. Thus,

$$L_f(P) = (m_1 + m_2 + \cdots + m_n)\Delta x, \qquad U_f(P) = (M_1 + M_2 + \cdots + M_n)\Delta x.$$

But f is continuous and decreasing. Therefore $m_i = f(x_i)$ and $M_i = f(x_{i-1})$. Thus

$$\begin{aligned} U_f(P) - L_f(P) &= \Delta x\big[\big(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\big) - \big(f(x_1) + f(x_2) + \cdots + f(x_n)\big)\big] \\ &= \Delta x \cdot \big(f(x_0) - f(x_n)\big) = \Delta x \cdot \big(f(a) - f(b)\big). \end{aligned}$$

38. (a) Given $f(x) = x^2$ and P is the regular partition of $[0,b]$, then $m_i = f(x_{i-1})$ since f is increasing. Also, $\Delta x_i = \frac{b}{n}$ and $x_i = \frac{ib}{n}$ for all i. Therefore,

$$\begin{aligned} L_f(P) &= \frac{b}{n}\left(f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right) = \frac{b}{n}\left[0^2 + \left(\frac{b}{n}\right)^2 + \left(\frac{2b}{n}\right)^2 + \cdots + \left(\frac{(n-1)b}{n}\right)^2\right] \\ &= \frac{b^3}{n^3}\left[1^2 + 2^2 + \cdots + (n-1)^2\right]. \end{aligned}$$

(b) Similar to part (a), we get

$$\begin{aligned} U_f(P) &= \frac{b}{n}\left(f(x_1) + f(x_2) + \cdots + f(x_n)\right) = \frac{b}{n}\left[\left(\frac{b}{n}\right)^2 + \left(\frac{2b}{n}\right)^2 + \cdots + \left(\frac{nb}{n}\right)^2\right] \\ &= \frac{b^3}{n^3}\left[1^2 + 2^2 + \cdots + n^2\right]. \end{aligned}$$

(c) Using the result from question 38, we have

$$L_f(P) = \frac{b^3}{n^3}\frac{n(n-1)(2n-1)}{6}, \qquad U_f(P) = \frac{b^3}{n^3}\frac{n(n+1)(2n+1)}{6}.$$

Applying limits,

$$\lim_{n\to\infty} L_f(P) = \lim_{n\to\infty}\frac{n(n-1)(2n-1)}{6n^3} = \frac{1}{3}, \qquad \lim_{n\to\infty} U_f(P) = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3},$$

so $\displaystyle\int_0^b x^2\,dx = \frac{1}{3}b^3$.

2 (SHE 5.3)

12. If $F(x) = \int_2^x (t+1)^3 \, dt$, then $F'(x) = (x+1)^3$. So $F'(-1) = 0$, $F'(0) = 1$, $F'(\frac{1}{2}) = \frac{27}{8}$, and $F''(x) = 3(x+1)^2$.

26. Let $f(x) = \int_1^x \sqrt{1-t^2} \, dt$ and $g(x) = \cos x$. Then we must differentiate $F(x) = f(g(x))$, which by the Chain Rule is $f'(g(x)) \cdot g'(x)$. Hence

$$F'(x) = \sqrt{1-\cos^2 x} \cdot (-\sin x) = -\sqrt{\sin^2 x} \sin x = -|\sin x| \sin x.$$

36. Suppose $\int_0^x \left[t \int_1^t f(u) \, du \right] dt$. Then $F'(x) = x \int_1^x f(u) \, du$. Hence $F'(1) = 1 \cdot 0 = 0$. Differentiating again we have (by the product rule) $F''(x) = \int_1^x f(u) \, du + x f(x)$, so $F''(1) = 0 + f(1) = f(1)$.

3 (SHE 5.4)

16. $\int_0^a (a^2 x - x^3) \, dx = \left[\frac{a^2 x^2}{2} - \frac{x^4}{4} \right]_0^a = \frac{a^4}{4}$.

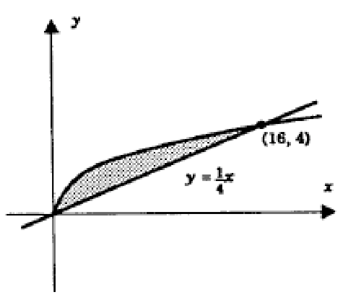
34. $\int_0^{\frac{\pi}{2}} \left[\frac{d}{dx} \sin^3 x \right] dx = \left[\sin^3 x \right]_0^{\frac{\pi}{2}} = 1$.

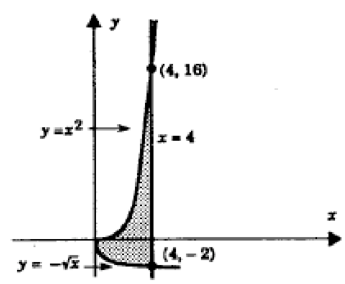
46. Note that $\int_{-4}^2 (2x+3) \, dx = \left[x^2 + 3x \right]_{-4}^2 = 6$, but

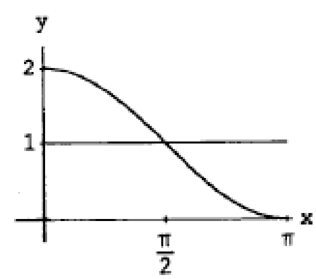
$$\int_{-4}^2 |2x+3| \, dx = \int_{-4}^{-3/2} (-2x-3) \, dx + \int_{-3/2}^2 (2x+3) \, dx = \left[-x^2 - 3x \right]_{-4}^{-3/2} + \left[x^2 + 3x \right]_{-3/2}^2 = \frac{37}{2}.$$

2 (SHE 5.5)

16. $A = \int_0^{16} \sqrt{x} - \frac{1}{4} x \, dx = \frac{32}{3}$. The sketch is given below.







22. $A = \int_0^4 (x^2 + \sqrt{x}) \, dx = \frac{80}{3}$. The sketch is given above center.

36. The sketch is given above right. The region consists of two identical regions, so by symmetry,

$$A = 2 \int_0^{\pi/2} (1 + \cos x - 1) \, dx = 2 \sin x \Big|_0^{\pi/2} = 2.$$

5 (SHE 5.6)

28. We have $f'(x) = x - \frac{x^2}{2} + C$, but $f'(2) = 1$ so $f'(x) = x - \frac{x^2}{2} + 1$. Hence $f(x) = \frac{x^2}{2} - \frac{x^3}{6} + x + K$ for some constant K. But $f(2) = 0$, so $f(x) = -\frac{x^3}{6} + \frac{x^2}{2} + x - \frac{8}{3}$.

6 Let $F(x) = \int_0^x f(u)(x-u)\,du = \int_0^x xf(u) - uf(u)\,du = x\int_0^x f(u)\,du - \int_0^x uf(u)\,du$ and $G(x) = \int_0^x \left(\int_0^u f(t)\,dt\right) du$. By FTC and the product rule, $F'(x) = xf(x) + \int_0^x f(u)\,du - xf(x) = \int_0^x f(u)\,du$; and $G'(x) = \int_0^x f(t)\,dt$. By inspection $F'(x) = G'(x)$, so it follows that $F(x) = G(x) + C$. But clearly $F(0) = G(0) = 0$, so it follows that $C = 0$, so $F(x) = G(x)$, as required.

7 (a) Let d be the distance, v be the velocity, and a be the acceleration. We have the properties that $d'(t) = v(t)$, and $v'(t) = a(t)$. When the train is accelerating, we have $v'(t) = a(t) = \frac{2}{3}$, so $v(t) = \frac{2}{3}t + v_0$. Since $v_0 = 0$, we can solve for the time it takes to accelerate to maximum speed: $120 = \frac{2}{3}t \implies t = 180$ seconds. The distance traveled while accelerating is $d(t) = v_0 t + \frac{1}{2}at^2$; since $v_0 = 0$ and $a = \frac{2}{3}$, we have $d(t) = \frac{1}{3}t^2$, so $d(180) = 10800$ meters. By symmetry, the train accelerates and decelerates for 180 seconds and 10800 meters. Thus, the train must be at maximum speed for exactly 90 seconds. The distance traveled at top speed is $d(90\text{ sec}) = 120\text{ m/sec} \cdot 90\text{ sec} = 10800$ meters.

Therefore, the distance between the airport and the city centre station is 32400 meters.

(b) Using the information in part (a), the train accelerates for 10800 meters and decelerates for 10800 meters. Therefore, the train between Shanghai and Hangzhou will stay at top speed for 148400 meters. The train stays at top speed for $\dfrac{148400\text{ m}}{120\text{ m/sec}} = 1236\dfrac{2}{3}$ seconds. Thus the total time taken from Shanghai and Hangzhou is $180 + 1236\frac{2}{3} + 180 = 1596\frac{2}{3}$ seconds, or 26 minutes and $36\frac{2}{3}$ seconds.

8 (SHE 5.7)

12. Let $u = 6 - 5s^2$. Then $du = -10s\,ds$ and

$$\int \frac{2s}{\sqrt[3]{6-5s^2}}\,ds = -\frac{1}{5}\int \frac{du}{\sqrt[3]{u}} = -\frac{3}{10}u^{2/3} + C = -\frac{3}{10}(6-5s^2)^{2/3} + C.$$

18. Let $u = x^2 + 3x + 1$. Then $du = (2x+3)\,dx$. Therefore,

$$\int \frac{4x+6}{\sqrt{x^2+3x+1}}\,dx = 2\int \frac{du}{\sqrt{u}} = 4\sqrt{u} + C = 4\sqrt{x^2+3x+1} + C.$$

22. Let $u = 4 + 2x^3$. Then $du = 6x^2\,dx$. Changing the limits of integration, if $x = -1$ then $u = 2$ and if $x = 0$ then $u = 4$. Therefore,

$$\int_{-1}^{0} 3x^2(4+2x^3)^2\,dx = \frac{1}{2}\int_2^4 u^2\,du = \left[\frac{1}{6}u^3\right]_2^4 = \frac{28}{3}.$$

30. Let $u = (x+2)(x+3) = x^2 + 5x + 6$. Then $du = (2x+5)\,dx$. Changing the limits of integration, we have $x = 0 \implies u = 6$ and $x = 1 \implies u = 12$. Hence,

$$\int_0^1 \frac{2x+5}{(x+2)^2(x+3)^2}\,dx = \int_6^{12} \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]_6^{12} = \frac{1}{12}.$$

48. Let $u = x^2$. Then $du = 2x\,dx$, so

$$\int x \sec^2 x^2 \, dx = \frac{1}{2} \int \sec^2 u \, du = \frac{1}{2} \tan u + C \implies \frac{1}{2} \tan x^2 + C.$$

72. Using the formula $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, we have

$$\int \cos^2 x \, dx = \int \frac{1}{2} + \frac{1}{2} \cos 2x \, dx = \frac{1}{2} x + \frac{1}{4} \cos 2x + C.$$