MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 7

1 (SHE 3.7)

18. Differentiating, $x^2+4xy+y^3+5=0 \implies 2x+4y+4x\frac{dy}{dx}+3y^2\frac{dy}{dx}=0$. At $(2,-1)$ we get $4-4+8\frac{dy}{dx}+3\frac{dy}{dx}=0$, so $\frac{dy}{dx}=0$. Differentiating again, we get
$$2+4\frac{dy}{dx}+4\frac{dy}{dx}+4x\frac{d^2y}{dx^2}+6y\left(\frac{dy}{dx}\right)^2+3y^2\frac{d^2y}{dx^2}=0.$$
At $(2,-1)$ we get $2+11\frac{d^2y}{dx^2}=0$, so $\frac{d^2y}{dx^2}=-\frac{2}{11}$.

46. For $y=2x$, the slope is $m_1=2$. For $x^2-xy+2y^2=28$, we have
$$2x-y-x\frac{dy}{dx}+4y\frac{dy}{dx}=0 \implies \frac{dy}{dx}=m_2=\frac{y-2x}{4y-x}.$$
At a point of intersection of the line and the curve we have $m_2=0$ since $y=2x$. Thus $\tan\alpha=|-m_1|=2$, so α is the angle between 0 and $\frac{\pi}{2}$ such that $\tan\alpha=2$.

56. (a) Differentiating, $x^{2/3}+y^{2/3}=a^{2/3} \implies \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$, so $\frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$. Thus the slope at (x_1,y_1) (where $x_1\neq 0$) is $m=-\left(\frac{y_1}{x_1}\right)^{1/3}$.

(b) From part (a), it is easy to see that if $m=0$, then $y_1=0$, so the points for which $m=0$ are $(a,0)$ and $(-a,0)$.
For $m=1$, we solve $-(y_1/x_1)^{1/3}=1$. Hence $y_1=-x_1$. This yields $x_1=\pm\frac{1}{4}a\sqrt{2}$. Hence the points for which $m=1$ are $(\frac{1}{4}a\sqrt{2},-\frac{1}{4}a\sqrt{2})$ and $(-\frac{1}{4}a\sqrt{2},\frac{1}{4}a\sqrt{2})$.
For $m=-1$, we solve $-(y_1/x_1)^{1/3}=-1$. Hence $y_1=x_1$. This yields $x_1=\pm\frac{1}{4}a\sqrt{2}$. Hence the points for which $m=1$ are $(-\frac{1}{4}a\sqrt{2},-\frac{1}{4}a\sqrt{2})$ and $(\frac{1}{4}a\sqrt{2},\frac{1}{4}a\sqrt{2})$.

2 (i) The circles $x^2+y^2=ax$ and $x^2+y^2=by$ intersect at the origin where the tangent lines are vertical and horizontal, respectively. If (x_0,y_0) is the other point of intersection, then $x_0^2+y_0^2=ax_0$ (*) and $x_0^2+y_0^2=by_0$ (**). Now $x^2+y^2=ax \implies 2x+2yy'=a \implies y'=\frac{a-2x}{2y}$ and $x^2+y^2=by \implies 2x+2yy'=by' \implies y'=\frac{2x}{b-2y}$. Thus, the curves are orthogonal at (x_0,y_0) if and only if
$$\frac{a-2x_0}{2y_0}=-\frac{b-2y_0}{2x_0} \iff 2ax_0-4x_0^2=4y_0^2-2by_0 \iff ax_0+by_0=2(x_0^2+y_0^2),$$
which is true by (*) and (**).

(ii) $y=ax^3 \implies y'=3ax^2$ and $x^2+3y^2=b \implies 2x+6yy'=0 \implies 3yy'=-x$. Hence,
$$y'=-\frac{x}{3y}=-\frac{x}{3(ax^3)}=-\frac{1}{3ax^2},$$
so the curves are orthogonal.

3 (SHE 4.10)

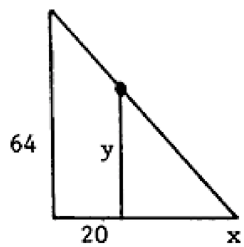
20. We use the diagram illustrated below left. Expressing y as a function of t, we have $y(t) = -16t^2$, or $\frac{dy}{dt} = -32t$. By similar triangles,

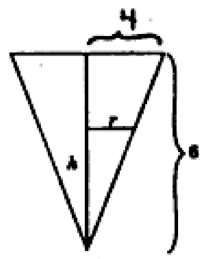
$$\frac{y}{x} = \frac{64}{20+x} \implies y = \frac{64x}{20+x}.$$

Differentiating,

$$\frac{dy}{dt} = \frac{(20+x)(64) - 64x}{(20+x)^2}\frac{dx}{dt} = \frac{1280}{(20+x)^2}\frac{dx}{dt} \implies \frac{dx}{dt} = \frac{(20+x)^2}{1280}\frac{dy}{dt}.$$

At $t = 1$, $y = 48$, $x = 60$, and $\frac{dy}{dt} = -32$, we get $\frac{dx}{dt} = \frac{80^2}{1280}(-32) = -160$ feet per second.





24. We use the diagram illustrated above. We have $V = \frac{1}{3}\pi r^2 h$ and by similar triangles, $\frac{r}{h} = \frac{4}{6}$. Therefore $V = \frac{4}{27}\pi h^3$. Thus $\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}$. Hence, when $\frac{dh}{dt} = \frac{1}{2}$ and $h = 2$, we get $\frac{dV}{dt} = \frac{8\pi}{9}$ cubic feet per second.

4 Let x be the horizontal distance between the vertical and the kite. Then $\tan\theta = 30/x$. Differentiating,

$$\sec^2\theta \frac{d\theta}{dt} = -\frac{30}{x^2}\frac{dx}{dt}.$$

When $x = 60$, then $\tan\theta = \frac{1}{2} \implies \sec^2\theta = \tan^2\theta + 1 = \frac{1}{4} + 1 = \frac{5}{4}$. Therefore

$$\frac{5}{4}\frac{d\theta}{dt} = -\frac{30}{900}\cdot 2 \implies \frac{d\theta}{dt} = -\frac{2}{75},$$

so the angle is decreasing at a rate of $2/75$ radians per second.

5 The hour hand of a clock goes around once every 12 hours, or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians per hour. The minute hand goes around once an hour, or at the rate of 2π radians per hour. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at a rate of $\frac{d\theta}{dt} = \frac{\pi}{6} - 2\pi == \frac{11\pi}{6}$ radians per hour. Now we relate θ to ℓ, the distance between the tips of the hands. By the law of cosines, $\ell^2 = 4^2 + 8^2 - 2\cdot 4\cdot 8\cos\theta = 80 - 64\cos\theta$ (*). Differentiating with respect to t, we get $2\ell\frac{d\ell}{dt} = -64(-\sin\theta)\frac{d\theta}{dt}$. At 2:00, the angle between the two hands is one-sixth of the circle, that is $\frac{\pi}{3}$ radians. We use (*) to find ℓ at 2:00:

$$\ell = \sqrt{80 - 64\cos\frac{\pi}{3}} = \sqrt{48} = 4\sqrt{3}.$$

Substituting, we get $2\ell\frac{d\ell}{dt} = 64\sin\frac{\pi}{3}(-\frac{11\pi}{6})$, which implies

$$\frac{d\ell}{dt} = \frac{64(\frac{\sqrt{3}}{2})(-\frac{11\pi}{6})}{2\cdot 4\sqrt{3}} = -\frac{22\pi}{3},$$

so at 2:00, the distance between the tips of the hands is decreasing at a rate of $\frac{22\pi}{3}$ cm/hr.

6 (SHE 4.1)

2. For $f(x) = x^4 - 2x^2 - 8$, the function is a polynomial, hence it is continuous and differentiable everywhere. Furthermore, $f(-2) = f(2) = 0$, hence by Rolle's Theorem, there exists $c \in (-2,2)$ such that $f'(c) = 0$. We can find c explicitly: $f'(x) = 4x^3 - 4x = 4x(x^2-1)$, so there are three values of c in the interval $(-2,2)$ such that $f'(c) = 0$, namely, $c = -1, 0, 1$.

6. $f(x) = 3\sqrt{x} - 4x$ is continuous and differentiable for all $x > 0$. Hence, by the Mean Value Theorem, there exists $c \in (1,4)$ such that
$$f'(c) = \frac{f(4) - f(1)}{4-1} = \frac{-10-(-1)}{4-1} = -3.$$
We can find c explicitly: $f'(x) = \frac{3}{2\sqrt{x}} - 4$. Solving,
$$\frac{3}{2\sqrt{c}} - 4 = -3 \implies \frac{2\sqrt{c}}{3} = 1 \implies c = \frac{9}{4},$$
which is certainly in the interval $(1,4)$.

24. Set $P(x) = 6x^5 + 13x + 1$, which is continuous and differentiable for all x (since it is a polynomial). Since $P(-1) < 0$ and $P(1) > 0$, by IVT, the equation $P(x) = 0$ has at least one real root c_1. Suppose the equation has another real root c_2. Then by Rolle's Theorem, there exists c between c_1 and c_2 such that $P'(c) = 0$. But $P'(x) = 30x^4 + 13 > 0$ for all x, a contradiction. Therefore, $P(x) = 0$ has only one solution.

40. We prove the result for $h > 0$, since the proof for $h < 0$ is similar. If f is differentiable on $(x, x+h)$, it is continuous there and thus, by the hypothesis at x and $x+h$, is continuous on $[x, x+h]$. By the Mean Value Theorem, there exists $c \in (x, x+h)$ such that
$$\frac{f(x+h) - f(x)}{x+h-x} = f'(c) \implies f(x+h) - f(x) = f'(c)h.$$
Since c is between x and $x+h$, c can be written as $c = x + \theta h$, where $0 < \theta < 1$, thus completing the proof.

7 Suppose f is an odd function and differentiable everywhere. Then by the Mean Value Theorem, for any $b > 0$, there exists $c \in (-b, b)$ such that
$$f'(c) = \frac{f(b) - f(-b)}{b-(-b)} = \frac{f(b) + f(b)}{2b} = \frac{f(b)}{b},$$
since f is odd.

8 (SHE 4.2)

18. We take cases on x. By the definition of absolute value, we have
$$f(x) = \begin{cases} x^2 - x - 2, & x \le -1, \\ -x^2 + x + 2, & -1 < x < 2, \\ x^2 - x - 2, & x \ge 2. \end{cases} \implies f'(x) = \begin{cases} 2x - 1, & x < -1, \\ -2x + 1, & -1 < x < 2, \\ 2x - 1, & x > 2. \end{cases}$$
Therefore f increases on $(-1, \frac{1}{2}) \cup (2, \infty)$ and f decreases on $(-\infty, -1) \cup (\frac{1}{2}, 2)$.

48. The statement in part (a) is true: Let $x_1, x_2 \in [a,c]$, where $x_1 < x_2$. If $x_1, x_2 \in [a,b]$, or if $x_1, x_2 \in [b,c]$, then $f(x_1) > f(x_2)$. If $x_1 \in [a,b)$ and $x_2 \in [b,c]$, then $f(x_1) > f(b) \geq f(x_2)$. Therefore f decreases on $[a,c]$. The statement in part (b) is false: consider the function

$$f(x) = \begin{cases} -x+2, & x \leq 1, \\ -x+3, & x > 1. \end{cases}$$

56. (a) Consider the function $h(x) = f(x) - g(x)$. Then $h'(x) = f'(x) - g'(x) > 0$ on $(0,c)$, and h is increasing on $(0,c)$. Since $h(0) = f(0) - g(0) = 0$, it follows that $h(x) > 0$ on $(0,c)$. Thus, $f(x) > g(x)$ on $(0,c)$.

    (b) Again, let $h(x) = f(x) - g(x)$. Then h is increasing on $(-c,0)$ which implies that $h(x) < 0$ on this interval since $h(0) = 0$. Therefore, $f(x) < g(x)$ on $(-c,0)$.

60. Let $f(x) = x - x^3/6$ and $g(x) = \sin x$. Then $f'(x) = 1 - x^2/2$, $g'(x) = \cos x$, $f''(x) = -x$, $g''(x) = -\sin x$, $f'''(x) = -1$, and $g'''(x) = -\cos x$. Note that $f''(0) = g''(0) = 0$ and $f'''(x) \leq g'''(x)$ for all $x > 0$; hence, $f''(x) < g''(x)$ for all $x > 0$ by question 56. Furthermore, $f'(0) = g'(0)$ and $f''(x) < g''(x)$, so by question 56, we have $f'(x) < g'(x)$ for $x > 0$. Applying question 56 once more, we have $f(0) = g(0)$, so it follows that $f(x) < g(x)$ for all $x > 0$, thereby ending the proof.