Theorem. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers such that $\sum_{n=0}^{\infty} a_n$ converges to S, then $\int_0^\infty \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right) e^{-x}\, dx$ exists and is equal to S. In other words,

$$\int_0^\infty \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right) e^{-x}\, dx = \sum_{n=0}^{\infty} \int_0^\infty \frac{a_n}{n!} x^n e^{-x}\, dx$$

proof. Consider a function f defined on $(0, \infty)$ by

$$f(R) = \int_0^R e^{-x} \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right) dx.$$

Since the sum $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n e^{-x}$ converges uniformly on $[0, R]$, the order of summation and integration in $f(R)$ can be exchanged, giving $f(R) = \sum_{n=0}^{\infty} \int_0^R \frac{a_n}{n!} x^n e^{-x}\, dx$. Then one can show that

$$S - f(R) = \sum_{n=0}^{\infty} \int_R^\infty \frac{a_n}{n!} x^n e^{-x}\, dx = e^{-R} \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{R^k}{k!}\right) a_n$$

by applying integration by parts successively.

Let $s_n = \sum_{k=0}^{n} a_k$ be partial sum of a_n, with $s_{-1} = 0$. Using summation by parts, $S - f(R) = e^{-R} \sum_{n=0}^{\infty} (S - s_{n-1}) \frac{R^n}{n!}$. Since $s_n \to S$ as $n \to \infty$, for $\epsilon > 0$, we can choose N such that $|S - s_{n-1}| < \frac{\epsilon}{2}$ whenever $n \geq N$. Then

$$\begin{aligned}
|S - f(R)| &\leq e^{-R} \sum_{n=0}^{\infty} |S - s_{n-1}| \frac{R^n}{n!} \\
&\leq e^{-R} \left(\sum_{n=0}^{N} |S - s_{n-1}| \frac{R^n}{n!}\right) + e^{-R} \sum_{n=N+1}^{\infty} |S - s_{n-1}| \frac{R^n}{n!} \\
&\leq \max_{0 \leq n \leq N} \left\{\frac{R^n}{n!} e^{-R}\right\} \left(\sum_{n=0}^{N} |S - s_{n-1}|\right) + e^{-R} \sum_{n=N+1}^{\infty} \frac{\epsilon}{2} \frac{R^n}{n!} \\
&\leq \max_{0 \leq n \leq N} \left\{\frac{R^n}{n!} e^{-R}\right\} \left(\sum_{n=0}^{N} |S - s_{n-1}|\right) + \frac{\epsilon}{2} \qquad \cdots (*)
\end{aligned}$$

. Choosing K sufficiently large, we have $\max_{0 \leq n \leq N} \left\{\frac{R^n}{n!} e^{-R}\right\} \left(\sum_{n=0}^{N} |S - s_{n-1}|\right) < \frac{\epsilon}{2}$ for $R > K$. Together with $(*)$, we have $|S - f(R)| < \epsilon$ for sufficiently large R. Therefore, $\int_0^\infty e^{-x} \left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right) dx = \lim_{R \to \infty} f(R)$ exists and is equal to S as desired.