MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 6

1 (SHE 3.1)

10. Applying the definition of derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$
$$= \lim_{h\to 0}\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$
$$= \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = -\frac{1}{2x^{3/2}}$$

14. For $f(x) = 5 - x^4$ and $c = -1$, the difference quotient is

$$\frac{f(-1+h)-f(-1)}{h} = \frac{5-(-1+h)^4-4}{h} = \frac{5-(h^4-4h^3+6h^2-4h+1)-4}{h}$$
$$= -h^3+4h^2-6h+4,$$

so $f'(-1) = \lim_{h\to 0} -h^3+4h^2-6h+4 = 4$.

30. Checking the one-sided limits, we have

$$\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+}\frac{2(1+h)^3+1-3}{h} = \lim_{h\to 0^+} 6+6h+2h^2 = 6,$$

and

$$\lim_{h\to 0^-}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-}\frac{3(1+h)^2-3}{h} = \lim_{h\to 0^-}\frac{3(1+2h+h^2)-3}{h} = \lim_{h\to 0^-} 6+3h = 6,$$

so $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = 6$, which implies that $f'(1) = 6$.

32. Since $\lim_{x\to 3^-} f(x) = -\frac{9}{2}$ and $\lim_{x\to 3^+} f(x) = -9$, the function is not continuous at $x = 3$, so the function is also not differentiable at 3.

52. (a) Since

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h},$$

and $f(x)/x = 1$ if x is rational (and non-zero) and 0 if x is irrational, it follows that $\lim_{x\to 0}\frac{f(x)}{x}$ does not exist. Hence f is not differentiable at 0.

(b) Note that

$$\lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = \lim_{h\to 0}\frac{g(h)}{h} = \lim_{h\to 0} f(h) = 0,$$

so g is differentiable at 0 and $g'(0) = 0$.

2 (SHE 3.2)

24. $f'(x) = \dfrac{(x+1)^2(4x+1)-(2x^2+x+1)\cdot 2(x+1)}{(x+1)^4} = \dfrac{(x+1)(4x+1)-2(2x^2+x+1)}{(x+1)^3}$.
    Hence, $f'(0) = -1$, $f'(1) = \frac{1}{4}$.

42. Here, $f'(x) = \dfrac{(x^2+4)(2x-2)-(x^2-2x+4)(2x)}{(x^2+4)^2} = \dfrac{2(x^2-4)}{(x^2+4)^2}$.
    Therefore, $f'(x) = 0$ at $x = \pm 2$, $f'(x) > 0$ where $|x| > 2$ and $f'(x) < 0$ where $|x| < 2$.

60. Let (x_0, y_0) be the point on the graph that the tangent line passes through. $f'(x) = 3x^2$, so the slope of the tangent line is

$$3x_0^2 = \frac{y_0 - 8}{x_0 - 2} \implies 3x_0^2(x_0 - 2) = 3x_0^2 - 8 \implies x_0 = 2, -1.$$

Hence the lines are $y - 8 = 12(x-2)$ and $y + 1 = 3(x+1)$.

3 (SHE 3.3)

32. Simplifying, we have $f(x) = (4x^2 - 9)x^{-1} = 4x - 9x^{-1}$, hence $f'(x) = 4 + 9x^{-2}$ and $f''(x) = -18x^{-3}$.

40. Differentiating,

$$\begin{aligned}\frac{d^2}{dx^2}\left[(x^2-3x)\frac{d}{dx}(x+x^{-1})\right] &= \frac{d^2}{dx^2}[(x^2-3x)(1-x^{-2})] = \frac{d^2}{dx^2}[x^2-3x-1+3x^{-1}] \\ &= \frac{d}{dx}(2x-3-3x^{-2}) = 2+6x^{-3}.\end{aligned}$$

54. Since

$$\lim_{h\to 0^+} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^+} \frac{h^3 - 0}{h} = 0, \qquad \lim_{h\to 0^-} \frac{g(0+h)-g(0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0,$$

it follows that g is differentiable and $g'(0) = 0$. Hence $g'(x) = \begin{cases} 3x^2, & x > 0, \\ 0, x \le 0. \end{cases}$

Similarly, since

$$\lim_{h\to 0^+} \frac{g'(0+h)-g'(0)}{h} = \lim_{h\to 0^+} \frac{3h^2 - 0}{h} = 0,$$

and

$$\lim_{h\to 0^-} \frac{g'(0+h)-g'(0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0,$$

Then g' is differentiable and $g''(0) = 0$. Hence $g''(x) = \begin{cases} 6x, & x \ge 0, \\ 0, & x < 0. \end{cases}$.

This solves parts (a) and (b). For part (c), note that

$$\lim_{h\to 0^+} \frac{g''(0+h)-g''(0)}{h} = \lim_{h\to 0^+} \frac{6h - 0}{h} = 6,$$

but

$$\lim_{h\to 0^-} \frac{g''(0+h)-g''(0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0,$$

so g'' is not differentiable at 0.

6 (SHE 3.4)

16. Set $k(x) = f(x)g(x)h(x)$. Then $k'(x) = f(x)g(x)h'() + f(x)g'(x)h(x) + f'(x)g(x)h(x)$, so $k'(1) = 0 \cdot 2 \cdot 0 + 0 \cdot (-1) \cdot (-2) + 1 \cdot 2 \cdot (-2) = -4$.

7 (SHE 3.5)

20. Differentiating,

$$\begin{aligned} f'(x) &= 3[(2x+1)^2 + (x+1)^2]^2 \frac{d}{dx}[(2x+1)^2 + (x+1)^2] \\ &= 3[(2x+1)^2 + (x+1)^2]^2[2(2x+1)(2) + 2(x+1)(1)] \\ &= 6[(2x+1)^2 + (x+1)^2]^2(5x+3). \end{aligned}$$

36. $(f \circ h \circ g)'(1) = f'(h(g(1)))h'(g(1))g'(1) = f'(2)h'(1)g'(1) = 0$.

68. We are given that $V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$, and $\frac{dV}{dt} = 200$, where V is the volume and S is the surface area. By the chain rule,

$$\frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt} = 8\pi r \cdot \frac{1}{4\pi r^2} \cdot 200 = \frac{400}{r}.$$

So when $r = 5$, then $\frac{dS}{dt} = 80$, so the surface area is increasing 80 square centimetres per second at the instant the radius is 5 centimetres.

8 (SHE 3.6)

44. $\dfrac{dy}{dx} = -3\csc^2 x + 4$, so solving $\csc = \pm\frac{2}{\sqrt{3}}$ gives $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

54. Differentiating using (3.5.4), we get

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dx}\frac{dx}{dt} = (-2u)(-\csc x \cot x)(3) = -2\csc 3t(-\csc 3t \cot 3t)(3) = 6\csc^2(3t)\cot(3t).$$

Doing it the other way, we get

$$y = 1 - \csc^2(3t) \implies y' = -2\csc 3t(-\csc 3t \cot 3t)(3) = 6\csc^2 3t \cot 3t.$$

70(a). Setting right and left-hand derivatives equal to each other at $x = \frac{\pi}{3}$ yields $-a\sin\frac{\pi}{3} = \frac{1}{2}\cos\frac{\pi}{6}$, so $a = -\frac{1}{2}$. Setting right and left-hand values of f equal to each other at $x = \frac{\pi}{3}$ yields $1 - \frac{1}{2}\cos\frac{\pi}{3} = b + \sin\frac{\pi}{3}$, so $b = \frac{1}{4}$.