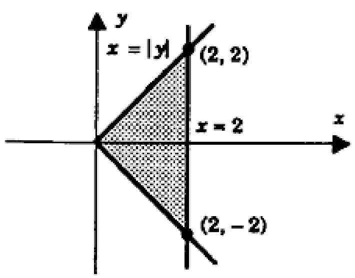
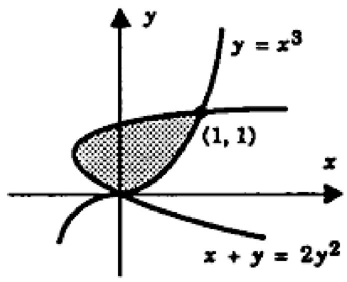
MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 10

1 (SHE 6.1)

10. See the figure below left. The area of the region ($A_x = A_y$) is

$$A_x = \int_0^2 (x-(-x))\,dx = \left[x^2\right]_0^2 = 4, \qquad A_y = \int_{-2}^0 [2-(-y)]\,dy + \int_0^2 (2-y)\,dy.$$



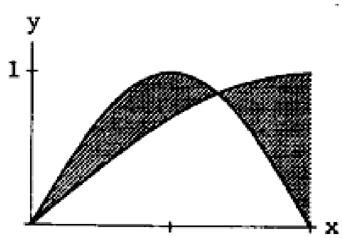


18. See the figure above right. The curves intersect when

$$\begin{aligned} x+x^3 = 2x^6 &\implies 2x^6 - x^3 - x = 0 \implies x(2x^5 - x^2 - 1) = 0 \\ &\implies x(x-1)(2x^4+2x^3+2x^2+x+1) = 0, \end{aligned}$$

or $x = 0, 1$. Hence the points of intersection are $(0,0)$ and $(1,1)$. The area of the region is

$$A = \int_0^1 y^{1/3} - (2y^2 - y)\,dy = \int_0^1 y^{1/3} + y - 2y^2\,dy = \left[\frac{3}{4}y^{4/3} + \frac{y^2}{2} - \frac{2}{3}y^3\right]_0^1 = \frac{7}{12}.$$



24. See the diagram above. The curves intersect when

$$\sin x = \sin 2x \implies \sin x = 2\sin x\cos x \implies x = 0, \frac{\pi}{3}.$$

The area is given by

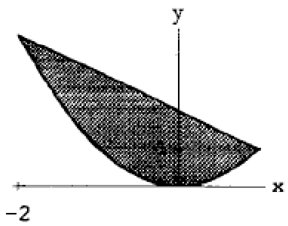
$$\begin{aligned} A &= \int_0^{\pi/3} \sin 2x - \sin x\,dx + \int_{\pi/3}^{\pi/2} \sin x - \sin 2x\,dx \\ &= \left[-\frac{\cos 2x}{2} + \cos x\right]_0^{\pi/3} + \left[-\cos x + \frac{\cos 2x}{2}\right]_{\pi/3}^{\pi/2} = \frac{1}{2}. \end{aligned}$$

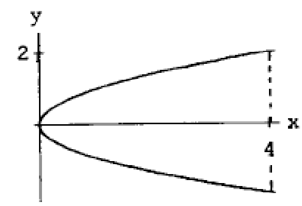
34. We want $\int_0^c \cos x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x\,dx$. This implies

$$\Big[\sin x\Big]_0^c = \frac{1}{2}\Big[\sin x\Big]_0^{\frac{\pi}{2}} = \frac{1}{2} \implies \sin c = \frac{1}{2} \implies c = \frac{\pi}{6}.$$

44. (a) The area of the region from $x=1$ to $x=b$ is $A = \int_1^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^b = 1-\frac{1}{b}$.

(b) Since $\lim_{b\to\infty} A(b) = 1$, so as $b\to\infty$, the area is finite.





2 (SHE 6.2)

10. The sketch of the region is given above left. The volume is

$$V = \int_{-2}^{1} \pi\left[(2-x)^2 - x^4\right]\,dx = \pi\left[-\frac{(2-x)^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{72\pi}{5}.$$

20. The sketch of the region is given above right. The volume is

$$V = \int_{-2}^{2} \pi[4^2 - y^4]\,dy = \pi\left[16y - \frac{y^5}{5}\right]_{-2}^{2} = \frac{256}{5}\pi.$$

26. The region bounded by $x = \sqrt{9-y^2}$ and $x=0$ is a half-sphere of radius 3, which is simply $\frac{2}{3}\pi\cdot 9 = 18\pi$. Alternatively, the volume is

$$V = \int_0^3 \pi(9-y^2)\,dy = \pi\left[9y - \frac{y^3}{3}\right]_0^3 = 18\pi.$$

30. (a) Volume with cross sections as rectangles of height h is

$$V = \int_0^1 2\sqrt{x}h\,dx + \int_1^3 2\cdot\frac{1}{\sqrt{2}}\sqrt{3-x}h\,dx = \frac{4h}{3}\left[x^{3/2}\right]_0^1 + \left[-\frac{2\sqrt{2}h}{3}(3-x)^{3/2}\right]_1^3 = 4h.$$

(b) Volume with cross sections as equilateral triangles is

$$\begin{aligned} V &= \int_0^1 \left(\frac{1}{2}\cdot 2\sqrt{x}\cdot\sqrt{3}\sqrt{x}\right)dx + \int_1^3 \left(\frac{1}{2}\cdot\sqrt{2}\sqrt{3-x}\cdot\frac{\sqrt{3}}{\sqrt{2}}\sqrt{3-x}\right)dx \\ &= \sqrt{3}\int_0^1 x\,dx + \frac{\sqrt{3}}{2}\int_1^3 (3-x)\,dx = \frac{3\sqrt{3}}{2}. \end{aligned}$$

(c) Volume with cross sections as isoceles triangles with the hypotenuse on the xy-plane:

$$\begin{aligned} V &= \int_0^1 \left(\frac{1}{2}\cdot 2\sqrt{x}\cdot\sqrt{x}\right) dx + \int_1^3 \left(\frac{1}{2}\cdot\sqrt{2}\sqrt{3-x}\cdot\frac{\sqrt{2}}{2}\cdot\sqrt{3-x}\right) dx \\ &= \int_0^1 x\,dx + \frac{1}{2}\int_1^3 (3-x)\,dx = \frac{3}{2}. \end{aligned}$$

38. When the ellipse $b^2x^2+a^2y^2=a^2b^2$ is revolved about the y-axis, the volume is

$$V = \int_{-b}^{b} \pi\left(\frac{a}{b}\sqrt{b^2-y^2}\right)^2 dy = \frac{\pi a^2}{b^2}\int_{-b}^{b}(b^2-y^2)\,dy = \frac{\pi a^2}{b^2}\left[b^2y-\frac{y^3}{3}\right]_{-b}^{b} = \frac{4}{3}\pi a^2 b.$$

3 (SHE 6.3)

6. The sketch is omitted. The volume is

$$V = \int_0^1 2\pi x[x^{1/3}-x^2]\,dx = 2\pi\int_0^1 (x^{4/3}-x^3)\,dx = 2\pi\left[\frac{3}{7}x^{7/3}-\frac{x^4}{4}\right]_0^1 = \frac{5\pi}{14}.$$

8. The volume of the solid when the triangle is rotated about the y-axis is

$$\begin{aligned} V &= \int_1^3 2\pi x(x-1)\,dx + \int_3^5 2\pi x(6-x-1)\,dx = 2\pi\left(\int_1^3 x^2-x\,dx + \int_3^5 5x-x^2\,dx\right) \\ &= 2\pi\left(\left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^3 + \left[\frac{5x^2}{2}-\frac{x^3}{3}\right]_3^5\right) = 24\pi. \end{aligned}$$

24. The volume is

$$\begin{aligned} V &= \int_0^1 2\pi y\cdot 2\sqrt{y}\,dy + \int_1^2 2\pi y\cdot 2(2-y)\,dy = 4\pi\left(\int_0^1 y^{3/2}\,dy + \int_1^2 (2y-y^2)\,dy\right) \\ &= 4\pi\left(\left[\frac{2}{5}y^{5/2}\right]_0^1 + \left[y^2-\frac{y^3}{3}\right]_1^2\right) = \frac{64\pi}{15}. \end{aligned}$$

4 (i) Let $y_c=-x^2-3x+6$ and $y_l=3-x$. The curves intersect when

$$-x^2-3x+6=3-x \implies x^2+2x-3=0 \implies x=-3,1.$$

Then

$$V = \pi\int_{-3}^{1}(y_c)^2-(y_l)^2\,dx = \pi\int_{-3}^{1} x^4+6x^3-4x^2-30x+27\,dx = \frac{1792\pi}{15}.$$

(ii) Let y_c and y_l be defined as before. Then

$$V = 2\pi\int_{-3}^{1}(y_c-y_l)(3-x)\,dx = 2\pi\int_{-3}^{1}(x^3-x^2-9x+9)\,dx = \frac{256\pi}{3}.$$

(iii) The curves intersect where $-x^2-3x+6=3-x$, or $x^2+2x-3=0$, or $x=1,-3$. Therefore the volume is

$$\begin{aligned}\int_{-3}^{1} 2\pi(3-x)(-x^2-3x+6-(3-x))\,dx &= 2\pi\int_{-3}^{1} x^3-x^2-9x+9\,dx\\ &= 2\pi\left[\frac{1}{4}x^4-\frac{1}{3}x^3-\frac{9}{2}x^2+9x\right]_{-3}^{1} = \frac{256\pi}{3}.\end{aligned}$$

5 (SHE 7.1)

44. Given $f(x)=x-\pi+\cos x$, then $f'(x)=1-\sin x\geq 0$ for all x (and equal to zero only at one point), so the function is continuous and strictly increasing. Thus, the function is one-to-one, and therefore an inverse exists. Hence,

$$(f^{-1})'(-1)=\frac{1}{f'(f^{-1}(-1))}=\frac{1}{f'(\pi)}=1.$$

since $f'(\pi)=1-\sin\pi=1$.

52. Since $f'(x)=2\sqrt{16+(2x)^4}>0$ for all x, then f is increasing for all x, so f is one-to-one, which implies f has an inverse. For part (b), notice that $f(\frac{1}{2})=0$, so $f^{-1}(0)=\frac{1}{2}$. Thus, $(f^{-1})'(0)=\dfrac{1}{f'(\frac{1}{2})}=\dfrac{1}{2\sqrt{17}}=\dfrac{\sqrt{17}}{34}$.

6 (SHE 7.2)

20. $\frac{1}{2}\ln x=\ln\sqrt{x}$, so for the equation to hold we must have $\sqrt{x}=2x-1$. Solving this equation by squaring both sides gives us $x=4x^2-4x+1 \implies 4x^2-5x+1=0 \implies (4x-1)(x-1)=0 \implies x=\frac{1}{4},1$. However, $\ln(2x-1)$ is not defined for $x=\frac{1}{4}$, so the only solution is $x=1$.

24. The question is not posed correctly and therefore is omitted.

7 (SHE 7.3)

24. Let $u=\ln(x+a)$. Then $du=\dfrac{dx}{x+a}$, so $\displaystyle\int\frac{\ln(x+a)}{x+a}\,dx=\int u\,du=\frac{1}{2}u^2=\frac{1}{2}\ln^2(x+a)+C$.

26. Let $u=4-\tan 2x$. Then $du=-2\sec^2 2x\,dx$. Thus

$$\int\frac{\sec^2 2x}{4-\tan 2x}\,dx=-\frac{1}{2}\int\frac{du}{u}=-\frac{1}{2}\ln|u|+C=-\frac{1}{2}\ln|4-\tan 2x|+C.$$

36. Multiplying the integrand, we have

$$\int(3-\csc x)^2\,dx=\int 9-6\csc x+\csc^2 x\,dx=9x-6\ln|\csc x-\cot x|-\cot x+C.$$

50. Given $g(x)=x(x+a)(x+b)(x+c)$, we take the logarithm of both sides to obtain

$$\ln g(x)=\ln[x(x+a)(x+b)(x+c)]=\ln x+\ln(x+a)+\ln(x+b)+\ln(x+c).$$

Differentiating both sides gives us $\dfrac{1}{g(x)}\cdot g'(x)=\dfrac{1}{x}+\dfrac{1}{x+a}+\dfrac{1}{x+b}+\dfrac{1}{x+c}$, so

$$g'(x)=x(x+a)(x+b)(x+c)\left[\frac{1}{x}+\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}\right].$$

8 (i) Applying L'Hôpital's Rule,

$$\lim_{x\to\infty}\frac{(\ln x)^4}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{4(\ln x)^3}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{12(\ln x)^2}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{24(\ln x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{24}{x} = 0,$$

so the limit is zero.

(ii) The limit is of the form $(0\cdot\infty)$, so we re-write the limit and get

$$\begin{aligned}\lim_{x\to 0^+}\sin x\cdot\ln x &= \lim_{x\to 0^+}\frac{\ln x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x} = \lim_{x\to 0^+}\frac{1}{-x\csc x\cot x} = \lim_{x\to 0^+}\frac{\sin^2 x}{-x\cos x}\\ &= \lim_{x\to 0^+} -\frac{\sin x}{x}\cdot\tan x = -1\cdot 0 = 0.\end{aligned}$$