MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 12

1 (SHE 11.7)

10. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{a\to0^+}\int_a^1 \frac{dx}{\sqrt{1-x}} = \lim_{a\to0^+}\left[-2\sqrt{1-x}\right]_a^1 = \lim_{a\to0^+} 0+2\sqrt{1-a} = 2$, so the integral converges to 2.

26. $\displaystyle\int_1^\infty \frac{x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{x}{(1+x^2)^2}\,dx = \lim_{b\to\infty}\left[\frac{-1}{2(1+x^2)}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{2(1+b^2)}+\frac{1}{4}\right] = \frac{1}{4}$, so the integral converges to $\frac{1}{4}$.

54. Since $\displaystyle 0 \le \int_\pi^\infty \frac{\sin^2 2x}{x^2}\,dx \le \int_\pi^\infty \frac{1}{x^2}\,dx$, and $\displaystyle\int_\pi^\infty \frac{1}{x^2}\,dx$ converges (10.7.1), it follows by the integral comparison test (10.7.2) that $\displaystyle\int_\pi^\infty \frac{\sin^2 2x}{x^2}\,dx$ also converges.

2 (a) We prove by induction on n that $\displaystyle\lim_{x\to0^+} x(\ln x)^n = 0$. For $n=1$, we have

$$\lim_{x\to0^+} x\ln x = \lim_{x\to0^+}\frac{\ln x}{1/x} \overset{\text{H}}{=} \lim_{x\to0^+}\frac{1/x}{-1/x^2} = \lim_{x\to0^+}(-x) = 0,$$

so the base case is true. Now suppose the statement is true for $n=k$, i.e. $\displaystyle\lim_{x\to0^+} x(\ln x)^k = 0$. Then

$$\lim_{x\to0^+} x(\ln x)^{k+1} = \lim_{x\to0^+}\frac{(\ln x)^{k+1}}{1/x} \overset{\text{H}}{=} \lim_{x\to0^+}\frac{(k+1)(\ln x)^k\cdot(1/x)}{-1/x^2} = -(k+1)\lim_{x\to0^+} x(\ln x)^k = 0,$$

so the statement is true for all positive integers n.

(b) Once again we apply induction to show that $\displaystyle\int_0^1 (\ln x)^n\,dx = (-1)^n n!$. If $n=1$, then

$$\int_0^1 \ln x\,dx = \lim_{a\to0^+}\int_a^1 \ln x\,dx = \lim_{a\to0^+}\left[x\ln x - x\right]_a^1 = \lim_{a\to0^+} -1-a\ln a + a = -1 = (-1)^1 1!$$

since $\displaystyle\lim_{a\to0^+} a\ln a = 0$ by part (a). This proves the base case. Now assume $\displaystyle\int_0^1 (\ln x)^k\,dx = (-1)^k k!$ for some positive integer k. Integrating by parts and applying the result fro part (a), we have

$$\begin{aligned}\int_0^1 (\ln x)^{k+1}\,dx &= \lim_{a\to0^+}\int_a^1 (\ln x)^{k+1}\,dx = \lim_{a\to0^+}\left[x(\ln x)^{k+1}\right]_a^1 - (k+1)\lim_{a\to0^+}\int_a^1 (\ln x)^k\,dx \\ &= \lim_{a\to0^+} a(\ln a)^{k+1} - (k+1)\cdot(-1)^k k! = 0 + (-1)^{k+1}(k+1)! = (-1)^{k+1}(k+1)!,\end{aligned}$$

so the statement is true for $n=k+1$, and hence the statement is true for all positive integers n.

3 (SHE 12.5)

4. Computing derivatives,

$$\begin{aligned} f(x) = \sec x &\implies f'(x) = \sec x\tan x \implies f''(x) = \sec x\tan^2 x + \sec^3 x \\ &\implies f'''(x) = \sec x\tan^3 x + 5\sec^3 x\tan x \\ &\implies f^{(4)}(x) = \sec x\tan^4 x + 18\sec^3 x\tan^2 x + 5\sec^5 x,\end{aligned}$$

so

$$f(0)=1, \quad f'(0)=0, \quad f''(0)=1, \quad f'''(0)=0, \quad f^{(4)}(0)=5.$$

Hence, the fourth Taylor polynomial for $\sec x$ is

$$P_4(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4=1+\frac{x^2}{2}+\frac{5x^4}{24}.$$

8. Computing derivatives, if $f(x)=x\cos(x^2)$, then

$$\begin{aligned} f'(x)=\cos(x^2)-2x^2\sin(x^2) &\implies f''(x)=-6x\sin(x^2)-4x^3\cos(x^2) \\ &\implies f'''(x)=-24x^2\cos(x^2)-6\sin(x^2)+8x^4\sin(x^2) \\ &\implies f^{(4)}(x)=-60x\cos(x^2)+80x^3\sin(x^2)+16x^5\cos(x^2) \\ &\implies f^{(5)}(x)=-60\cos(x^2)+360x^2\sin(x^2)+240x^4\cos(x^2)-32x^6\sin(x^2), \end{aligned}$$

so all derivatives at 0 are zero except $f'(0)=1$ and $f^{(5)}(0)=-60$. Hence, the fifth Taylor polynomial of $f(x)=x\cos(x^2)$ is

$$P_5(x)=f'(0)x+\frac{f^{(5)}(0)}{5!}x^5=x-\frac{1}{2}x^5.$$

Alternatively, since $P_{2,\cos x}(x)=1-\frac{x^2}{2}$, then $P_{2,\cos x}(x^2)=P_{4,\cos(x^2)}(x)=1-\frac{x^4}{2}$. Subsequently, $x\cdot P_{4,\cos(x^2)}(x)=P_{5,x\cos(x^2)}(x)=x-\frac{x^5}{2}$, which is identical to our answer above.

14. Consider the function $f(x)=\ln(1-x)$. Then it is easy to show by induction that $f^{(n)}(x)=-(n-1)!(1-x)^{-n}$ for all $n\geq 1$. Hence $f(0)=0$ and $f^{(n)}(0)=-(n-1)!$ for all $n\geq 1$. Hence,

$$P_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}x^k=0+\sum_{k=1}^{n}\frac{-(k-1)!}{k!}x^k=\sum_{k=1}^{n}\frac{-1}{k}x^k=\sum_{k=1}^{n}\frac{-x^k}{k}.$$

16. Let $f(x)=\cos bx$. It follows that if k is odd, then $|f^{(k)}(x)|=|b^k\sin bx|$, so $f^{(k)}(0)=0$. For the even derivatives, it is easy to check that

$$f(0)=1, \quad f''(0)=-b^2, \quad f^{(4)}(0)=b^4, \cdots \quad f^{(2m)}(0)=(-1)^m b^{2m}.$$

So if n is even, then $n=2m$ for some integer m, and

$$P_n(x)=\sum_{k=0}^{m}\frac{(-1)^k}{(2k)!}(bx)^{2k}.$$

4 (SHE 12.6)

22. Since we are given $|f^{(n)}(3)|\leq 3$ for all x, we choose $M=3$. Then $f(2)=P_n(2)+R_n(2)$, where

$$|R_n(2)|\leq\frac{3|2|^{n+1}}{(n+1)!}=\frac{3\cdot 2^{n+1}}{(n+1)!}.$$

For three decimal place accuracy, we wish to find n such that

$$\frac{3\cdot 2^{n+1}}{(n+1)!}<5\cdot 10^{-4} \iff \frac{(n+1)!}{3\cdot 2^{n+1}}<\frac{1}{5\cdot 10^{-4}}=2\cdot 10^3 \iff \frac{(n+1)!}{2^{n+1}}<6\cdot 10^3.$$

Using trial-and-error, we observe that

$$n=9 \implies \frac{(n+1)!}{2^{n+1}} = \frac{14175}{4} < 6000, \qquad n=10 \implies \frac{(n+1)!}{2^{n+1}} = \frac{155925}{8} > 6000,$$

so $n=10$ is the least integer which satisfies the inequality.

24. Again we have $M=3$. We wish to find x such that

$$|R_9(x)| \leq \frac{3|x|^{n+1}}{(n+1)!} = \frac{3|x|^{10}}{10!} < \frac{1}{20} \implies |x|^{10} < \frac{10!}{60} = 60480.$$

So the values of x must satisfy $x \in (-\sqrt[10]{60480}, \sqrt[10]{60480})$. For reference, $\sqrt[10]{60480} \approx 3.0072$.

5 (a) Suppose $P_{2n+1}(x)$ is the $2n+1$-st Taylor polynomial for $\sin x$. Then $\sin x = P_{2n+1}(x) + R_{2n+1}(x)$ and $\lim_{n\to\infty} R_{2n+1}(x) = 0$. It then follows that $\lim_{n\to\infty} R_{2n+1}(x^2) = 0$ and

$$\sin(x^2) = P_{2n+1}(x^2) + R_{2n+1}(x^2),$$

thus $P_{2n+1}(x^2)$ must be the $4n+2$-nd Taylor polynomial for $\sin(x^2)$.

(b) Using part (a), we have the $4n+2$-nd Taylor polynomial for $\sin(x^2)$

$$P_{4n+2,\sin(x^2)}(x) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots + (-1)^n \frac{x^{4n+2}}{(2n+1)!}.$$

Hence,

$$P_{4n+3,x\sin(x^2)}(x) = x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \cdots + (-1)^n \frac{x^{4n+3}}{(2n+1)!}.$$

Finally,

$$P_{4n+3,x^2+x\sin(x^2)}(x) = x^2 + x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \cdots + (-1)^n \frac{x^{4n+3}}{(2n+1)!}.$$

(The question contains a typo, you should have been asked for the $4n+3$-rd Taylor polynomial, although the solution is identical, since the coefficients for x^{4n+4} and x^{4n+5} must both be zero.)

(c) If $P_n(x)$ is the n-th Taylor polynomial for $f(x)$, then it follows that $f^{(k)}(0) = k! \cdot a_k$, where a_k is the coefficient for the x^k term. Notice that the coefficient for x^{2008} must be zero (since 2008 can't be written in the form x^{4n+3} for some integer n), hence $f^{(2008)}(0)$. However 2007 can be written in the form $4n+3$, so if $n=501$, then $f^{(2007)}(0) = 2007! \cdot \frac{(-1)^{501}}{1003!} = -\frac{2007!}{1003!}$.

6 (i) See Section 12.6 Example 7.

(ii) Consider the function $f(x) = e^x$. Then in particular, $f^{(n)}(x) = e^x$, so if we let $x = -1/2$, then we can choose $d=1$ and so $|f^{(n)}(x)| \leq e < 3$. Thus we let $M=3$ and thus

$$R_n(-\tfrac{1}{2}) \leq \frac{3 \cdot \frac{1}{2}^{n+1}}{(n+1)!} = \frac{3}{2^{n+1}(n+1)!} < \frac{1}{100000}.$$

Thus we require to find n such that $2^{n+1} \cdot (n+1)! > 300000$. Since $2^7 \cdot 7! = 645120 > 300000$ and $2^6 \cdot 6! = 46080 < 300000$, it follows that

$$P_6(-\tfrac{1}{2}) = 1 - \frac{1}{2} + \frac{1}{2^2(2!)} - \frac{1}{2^3(3!)} + \frac{1}{2^4(4!)} - \frac{1}{2^5(5!)} + \frac{1}{2^6(6!)}$$

estimates $e^{-1/2}$ to within 10^{-5}.

As a check, we can use a calculator to find that $P_6(-\frac{1}{2}) - e^{-1/2} \approx 0.14584 \cdot 10^{-5} < 10^{-5}$, so our estimate is well within our desired error of 10^{-5}.

(iii) Solution omitted. Using our form of reminder does not lead to an elegant solution, so let's pretend this question never existed, okay?

7 (a) Since M is finite and

$$|R_n(x)| \leq \frac{M|x|^{n+1}}{(n+1)!} \iff -\frac{M|x|^{n+1}}{(n+1)!} \leq R_n(x) \leq \frac{M|x|^{n+1}}{(n+1)!} \implies -\frac{M|x|}{(n+1)!} \leq \frac{R_n(x)}{x^n} \leq \frac{M|x|}{(n+1)!},$$

so by the squeeze theorem, it follows that $\lim_{x\to 0} \frac{R_n(x)}{x^n} = 0$.

(b) (i) Using the result of part (a), we have

$$\begin{aligned}\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} &= \lim_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2!} + R_2(x)\right) - 1 - x}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2} + R_2(x)}{x^2} = \lim_{x\to 0} \left(\frac{1}{2} + \frac{R_2(x)}{x^2}\right) \\ &= \frac{1}{2} + 0 = \frac{1}{2}.\end{aligned}$$

You can verify your answer by two simple applications of L'Hôpital's Rule.

(ii) Similar to our approach to part (i), let $R_n(x)$ be the associated remainder term for e^x and $Q_n(x)$ be the associated remainder term for $\sin x$. Then

$$\begin{aligned}\lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} - e^x}{\sin x - x} &= \lim_{x\to 0} \frac{1 + x + \frac{x^2}{2} - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + R_3(x)\right)}{\left(x - \frac{x^3}{6} + Q_3(x)\right) - x} = \lim_{x\to 0} \frac{-\frac{x^3}{6} - R_3(x)}{-\frac{x^3}{6} + Q_3(x)} \\ &= \lim_{x\to 0} \frac{-\frac{1}{6} - \frac{R_3(x)}{x^3}}{-\frac{1}{6} + \frac{Q_3(x)}{x^3}} = \frac{-\frac{1}{6} - 0}{-\frac{1}{6} + 0} = 1.\end{aligned}$$