Lec 5

More About Matrices :-

Let A be a $n \times m$ matrix. There are two integers associated with A. One is called rank $= \mathcal{R}(A)$ and the other is called nullity $= \mathcal{N}(A)$. They are defined in such a way that

$$\mathcal{R}(A) + \mathcal{N}(A) = m$$

Fact: For any matrix A the # of l.i. rows equals the # of l.i. columns. This number is called the rank of A.

Remark: By definition $\mathcal{R}(A) \leq \min(m, n)$.

Example 5·1 :-

$$A = \begin{pmatrix} 3 & 6 & 9 \\ 4 & 8 & 12 \end{pmatrix}$$

$$n = 2, \quad m = 3.$$

The vectors

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 9 \\ 12 \end{pmatrix}$$

are l.d. There exists only one l.i. column vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Likewise $(3 \quad 6 \quad 9)$ & $(4 \quad 8 \quad 12)$ are l.d.. There exists only one l.i. row vector $(3 \quad 6 \quad 9)$. Hence

$$\text{rank } A = 1$$

Def: The row space of A is the vector space spanned by the rows of A.

The column space of A is the vector space spanned by the columns of A.

The null space of A is the set of all vectors x in $\mathbb{R}^m$ such that $Ax = 0$.

Fact:

$\mathcal{R}(A)$ = dimension of the row space
= dimension of the column space.

$\mathcal{N}(A)$ = dimension of the null space of A.

Example 5·1 (continued):

$$\mathcal{N}(A) = m - \mathcal{R}(A) = 3 - 1 = 2.$$

column space $= \left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right]$

row space $= \left[\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}\right]$

To find the null space:

$$x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$Ax = 0$$

$$\left.\begin{array}{l} \Rightarrow 3a + 6b + 9c = 0 \\ \quad 4a + 8b + 12c = 0 \end{array}\right\} \Rightarrow a + 2b + 3c = 0$$

$$\Rightarrow a = -2b - 3c$$

$$x = \begin{pmatrix} -2b-3c \\ b \\ c \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} b + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} c$$

Null space $= \left[\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right]$

Determinant calculation

Example 5.2 :

$$A = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 2 & 5 \\ 0 & 1 & -3 \end{pmatrix} \leftarrow$$

det A is computed by expanding it with respect to any row or column. Let us pick the 1^{st} row.

$$\det A = 3 \begin{vmatrix} 2 & 5 \\ 1 & -3 \end{vmatrix} - 6 \begin{vmatrix} 1 & 5 \\ 0 & -3 \end{vmatrix} + 9 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$= 3(-11) - 6(-3) + 9(1)$$

$$= -33 + 18 + 9 = -6$$

Let us pick the 2nd column.

det A =

$$-6\begin{vmatrix}1 & 5\\ 0 & -3\end{vmatrix} + 2\begin{vmatrix}3 & 9\\ 0 & -3\end{vmatrix} - 1\begin{vmatrix}3 & 9\\ 1 & 5\end{vmatrix}$$

$$= -6(-3) + 2(-9) - 1(15-9)$$

$$= 18 - 18 - 6 = -6$$

Same as before.

Try some other row or column for practice.

Some simple facts about determinant

① Let A be any square matrix. If B is another matrix obtained by interchanging any row or column of A, then

$$\det A = -\det B.$$

Example 5.3:

$$A = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 2 & 5 \\ 0 & 1 & -3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 6 & 9 \\ 0 & 1 & -3 \end{pmatrix}$$

$$B = \begin{pmatrix} 9 & 6 & 3 \\ 5 & 2 & 1 \\ -3 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & -3 \\ 3 & 6 & 9 \end{pmatrix}$$

We know from Example 5.2 that $\det A = -6$. Hence $\det B = 6$, $\det C = 6$

$$\det D = -6$$

2. Let A be any matrix such that a row or a column in A is repeated then

$$\det A = 0$$

Why? : Construct a matrix B out of A by interchanging the row or column which is repeated. Clearly by assumption $B = A$. However we also have

$$\det A = -\det B$$

Hence $\det A = -\det B = -\det A$

$$\Rightarrow \det A = 0$$

Example 5.4

$$A = \begin{pmatrix} 3 & 6 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \det A = 0$$

③ If any row or column of A is all zeros then $\det A = 0$.

Example 5·5

$$A = \begin{pmatrix} 0 & 6 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \det A = 0$$

④ Let A and B be two square matrices that are equal except any one i^{th} row or column, Assume that this particular row or column in B is λ times the corresponding row or column in A respectively then

$$\det B = \lambda \det A$$

Example 5·6 :

$$A = \begin{pmatrix} 3 & 6 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 6 & 3 \\ 4 & 8 & 4 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det B = 4 \det A = 4(-6) = -24$$

(5) Let A be any square matrix. Assume that B is constructed from A by adding the i^{th} row/column of A to the j^{th} row/column of A respectively where $i \neq j$, then

$$\det A = \det B$$

<u>Example 5·7</u>:

$$A = \begin{pmatrix} 3 & 6 & 3 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 8 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix} \qquad \det A = \det B = \det C$$

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$$\det\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

$$= \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} +$$

$$\det\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The same can be done in any other row or column.

Some not so simple facts about determinant :

① If $AB = C$

where A, B, C are $n \times n$ matrices.

$$\det C = \det A \cdot \det B.$$

② If $C = A + B$

it is not true in general that.

$$\det C \overset{??}{=} \det A + \det B.$$

③ If $B = A^{-1}$

then

$$\det B = \frac{1}{\det A}.$$

A is invertible ie A^{-1} exists iff.

$$\det A \neq 0.$$

(4) If $B = A^T$

A is a $n \times n$ matrix then

$\det B = \det A$.

Def: A $n \times n$ matrix A is said to be invertible if $\exists$ another $n \times n$ matrix B such that $BA = AB = I$.

Such a matrix B is denoted by A^{-1}.

Note that $\det I = 1$. Hence

$$\det(AB) = \det A \cdot \det B = 1$$

$$\Rightarrow \det A = \frac{1}{\det B}.$$

Fact: If A is invertible then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Since $\det A \neq 0$, $\frac{1}{\det A}$ exists.

Less intuitive fact about determinant:

(1) Let
$$\underset{n\times n}{H} = \left(\begin{array}{c|c} \underset{m\times m}{A} & B \\ \hline C & \underset{p\times p}{D} \end{array}\right)$$

H is $n \times n$

A is $m \times m$ $\qquad m+p=n$

D is $p \times p$

If A is invertible then we can write

$$\det H = \det A \ \det\left(D - CA^{-1}B\right)$$

Example 5.8:

$$H = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 8 & 5 \\ 3 & 2 & 6 & 5 \\ 5 & 6 & 0 & 0 \end{pmatrix}$$

5.15

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad A^{-1} = A \qquad \det A = 1$$

$$\det H = \det(D - CB)$$

$$D = \begin{pmatrix} 6 & 5 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 2 \\ 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 8 & 5 \end{pmatrix}$$

$$CB = \begin{pmatrix} 6+16 & 3+10 \\ 10+48 & 5+30 \end{pmatrix} = \begin{pmatrix} 22 & 13 \\ 58 & 35 \end{pmatrix}$$

$$D - CB = \begin{pmatrix} -16 & -8 \\ -58 & -35 \end{pmatrix}$$

$$\det H = 16 \cdot 35 - 58 \cdot 8$$
$$= 560 - 464$$
$$= 96$$

(2) Let

$$H = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$m \times m$

$n \times n$

$p \times p$

If D is invertible then

$$\det H = \det D \det\left(A - BD^{-1}C\right)$$

If A and D are both singular that is they cannot be inverted bad luck.

(3) Let A be a $m \times n$ matrix
B be a $n \times m$ matrix $n > m$.

Define $C = AB$ so that C is a $m \times m$ matrix. Then

$$\det C = \sum_{i_1, \ldots, i_m = 1}^{n} \det A_{i_1 \cdots i_m} \cdot \det B_{i_1 \cdots i_m}$$

where $A_{i_1 \cdots i_m}$ is a $m \times m$ square matrix obtained by stacking i_j^{th} column of A as the j^{th} column for $j = 1, 2, \ldots, m$.

Example 5.9:

$$A = \begin{pmatrix} 3 & 5 & 1 \\ 2 & 6 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 6 \\ 5 & -1 \\ 2 & 4 \end{pmatrix}$$

$$C = AB$$

$\det C =$

$$\begin{vmatrix} 3 & 5 \\ 2 & 6 \end{vmatrix} \begin{vmatrix} 3 & 6 \\ 5 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 2 & 9 \end{vmatrix} \begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix}$$

$$+ \begin{vmatrix} 5 & 1 \\ 6 & 9 \end{vmatrix} \begin{vmatrix} 5 & -1 \\ 2 & 4 \end{vmatrix}$$

Example 5.10

$$A = \left(\begin{array}{c|cc} 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 4 & 3 & 1 \end{array}\right)$$

$$\det A = \det \left(\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \right)$$

$$= \det \left(\begin{pmatrix} 0 & 1 \\ 3 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = -1 + 1 = 0$$

<u>Fact</u>: If a matrix A is of rank r, determinant of all $p \times p$ minors of A are zero for $p > r$. Furthermore $\exists$ at least one $r \times r$ minor of non-zero determinant.

<u>Example 5.11</u>

$$A = \begin{pmatrix} 5 & 10 & 15 \\ 6 & 12 & 18 \\ 9 & 18 & 27 \end{pmatrix}$$

rank $A = 1$

$$A_{12} = \begin{pmatrix} 5 & 10 \\ 6 & 12 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 5 & 15 \\ 9 & 27 \end{pmatrix}$$

$$A_{23} = \begin{pmatrix} 12 & 18 \\ 18 & 27 \end{pmatrix}$$

These are 2x2 principal minors

$\begin{pmatrix} 10 & 15 \\ 18 & 27 \end{pmatrix}, \begin{pmatrix} 6 & 18 \\ 9 & 27 \end{pmatrix}$ etc are non-principal minors of A

check that the determinant of all these minors are zero.

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Rank 1 matrix:

Let A be a $n \times n$ rank 1 matrix. It can always be written as a product of a column vector and a row vector.

ie
$$A = b c^T$$

where

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Example 5.11 (continued)

$$A = \underbrace{\begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix}}_{b} \underbrace{\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}}_{c^T}$$

Ofcourse the choice of b and c are not unique

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Fact: Let $A = bc^T$ be any rank 1 matrix then

$$\text{trace } A = c^T b$$

In fact if M is any $n \times n$ matrix then

$$\boxed{\text{trace } M(bc^T) = c^T M b}$$

The story of $\det(\lambda I + A)$:-

For every square $n \times n$ matrix A $\exists$ a polynomial of degree n given by

$$\det(\lambda I + A) =$$

$$\lambda^n + \Delta_1 \lambda^{n-1} + \Delta_2 \lambda^{n-2} + \cdots + \Delta_n .$$

What is perhaps surprising is that

$\Delta_j =$ sum of determinants of all $j \times j$ principal minors of A.

In particular

$\Delta_1 = \operatorname{trace} A$

$\Delta_n = \det A$.

Example 5.12

consider matrix A in Example 5.2.

$$\lambda I + A = \begin{pmatrix} \lambda+3 & 6 & 9 \\ 1 & \lambda+2 & 5 \\ 0 & 1 & \lambda-3 \end{pmatrix}$$

$\det(\lambda I + A) =$

$$(\lambda+3)\det\begin{pmatrix} \lambda+2 & 5 \\ 1 & \lambda-3 \end{pmatrix}$$

$$-1\det\begin{pmatrix} 6 & 9 \\ 1 & \lambda-3 \end{pmatrix}$$

$$= (\lambda+3)\left[\lambda^2 + 2\lambda - 3\lambda - 6 - 5\right]$$

$$- \left[6\lambda - 18 - 9\right]$$

$$= (\lambda+3)(\lambda^2 - \lambda - 11) - (6\lambda - 27)$$

$$= \lambda^3 - \lambda^2 - 11\lambda$$
$$+3\lambda^2 - 3\lambda - 33$$
$$-6\lambda + 27$$

$$= \lambda^3 + \underset{\text{trace } A}{2}\lambda^2 - 20\lambda - \underset{\det A}{6}$$

What is -20 ??

It is $\begin{vmatrix} 3 & 6 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 9 \\ 0 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 1 & -3 \end{vmatrix}$

$$= 0 + (-9) + (-6-5)$$

$$= -20$$

Sum of all 2x2 principal minors of A.

Special case:

If we substitute $\lambda = 1$ in the identity

$$\det(\lambda I + A) = \lambda^n + \Delta_1 \lambda^{n-1} + \cdots + \Delta_n$$

we obtain

$$\det(I + A) = 1 + \Delta_1 + \cdots + \Delta_n .$$

If A is a rank 1 $n \times n$ matrix it follows that

$$\det(I + A) = 1 + \operatorname{trace} A$$

A is rank 1.

writing $A = bc^T$ we have

$$\det(I + bc^T) = 1 + c^T b .$$

Example 5·13

If B is an invertible matrix and if A is a rank 1 matrix we can write $A = bc^T$. It follows that

$$B + A = B\left(I + B^{-1} b c^T\right)$$

$$\det(B+A) = \det B \left(1 + c B^{-1} b\right)$$

Actually

$$\det(B+A) = \det B \det\left(I + B^{-1} b c^T\right)$$

$$= \det B \left(1 + \text{trace}\, \underbrace{B^{-1} b c^T}_{\text{rank 1 matrix.}}\right)$$

$$= \det B \left(1 + c^T B^{-1} b\right)$$

Since $B^{-1} = \operatorname{adj} B / \det B$, we have

$$\boxed{\det(B+A) = \det B + c \operatorname{adj} B\, b}$$

Rank 1 perturbation of the B matrix.

$\det(\lambda I + A)$ is close but not exactly the characteristic polynomial of A.

The characteristic polynomial $p(\lambda)$ is

$$p(\lambda) = \det(\lambda I - A)$$

$$= \lambda^n - \Delta_1 \lambda^{n-1} + \Delta_2 \lambda^{n-2} - \Delta_3 \lambda^{n-3} + \cdots$$
$$\cdots + (-1)^n \Delta_n .$$

Example 5.14:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\delta_3 & -\delta_2 & -\delta_1 \end{pmatrix}$$

In the notation of page 5.22 we have

$\Delta_1 = -\delta_1$

$\Delta_2 = \delta_2$

$\Delta_3 = -\delta_3$

Hence

$\det(\lambda I - A) =$

$\lambda^3 + \delta_1 \lambda^2 + \delta_2 \lambda + \delta_3$

Similarity Transformation

Let A be any $n \times n$ matrix

Let T be any $n \times n$ invertible matrix i.e. T^{-1} exists. We define

$B = T^{-1} A T$ called the similarity transformation of A.

Def: The matrices A and B are called similar if $\exists\, T : B = T^{-1} A T$.

Fact: The characteristic polynomials of two similar matrices are the same.

Let
$$p_1(\lambda) = \det(\lambda I - A) \quad \& \; B = T^{-1} A T$$
$$p_2(\lambda) = \det(\lambda I - B)$$

we have

$$\begin{aligned} p_2(\lambda) &= \det(\lambda I - T^{-1} A T) \\ &= \det\left(T^{-1}(\lambda I - A) T\right) \end{aligned}$$

$$= \det(T^{-1}) \det(\lambda I - A) \det T$$

$$= \frac{1}{\det T} p_1(\lambda) \det T = p_1(\lambda).$$

<u>Corollary</u>:

$\Delta_1, \Delta_2, \ldots, \Delta_n$ are all invariant under similarity transformation. In particular if

$$B = T^{-1} A T$$

then

$$\det B = \det A$$

$$\operatorname{trace} B = \operatorname{trace} A.$$

etc.