MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 8

1 (i) For $f(x) = (x^2-1)^3$, the derivative is $f'(x) = 6x(x^2-1)$, so the critical points are $x = 0, \pm 1$ and the endpoints are $x = -1, 2$. For the local extrema, we see that $f'(x) > 0$ when $x \in (-1,0) \cup (1,\infty)$ and $f'(x) < 0$ when $x \in (-\infty,-1) \cup (0,1)$. Hence local minima occur at the points $(-1,0)$ and $(1,0)$ and $(0,-1)$.

For absolute extrema we compare the values of the function at the endpoints and critical points. We have

$$f(-1) = 0, \quad f(0) = -1, \quad f(1) = 0, \quad f(2) = 27,$$

so the absolute maximum is $(2,27)$ and the absolute minima are $(-1,0)$ and $(1,0)$.

(ii) Given $f(x) = x - 2\cos x$, we have $f'(x) = 1 + 2\sin x > 0$, since $\sin x$ is positive on $[0,\pi]$. Therefore there are no local extrema. Looking at the endpoints, $f(0) = -2$ and $f(\frac{\pi}{3}) = \frac{2\pi - 3\sqrt{3}}{6} > 0$, so the absolute minimum is at $x = 0$ and the absolute maximum is at $x = \frac{\pi}{3}$.

2 (SHE 4.3)

38. Suppose $f(x) = Ax^2 + Bx + C$. Then $f'(x) = 2Ax + B = 0 \implies x = -B/2A = 2$, so $B = -4A$. Furthermore, $f(-1) = 3$ implies $A - B + C = 3$ and $f(3) = -1$ implies $9A + 3B + C = -1$. Solving for A, B, and C, we get $A = \frac{1}{2}$, $B = -2$, $C = \frac{1}{2}$.

3 (SHE 4.4)

36. Suppose $f(x) = (1+x)^r - (1+rx)$ for $x \geq -1$. Then $f'(x) = r[(1+x)^{r-1} - 1]$, so $f'(x) = 0$ when $x = 0$. Furthermore, $f''(x) = r(r-1)(1+x)^{r-2}$, so $f''(0) = r(r-1) > 0$, so f has a local minimum at $x = 0$. By Theorem 4.4.3, $f(0) = 0$ is the absolute minimum of f.

4 (SHE 4.5)

14. By doing question 13, we see that $A(m) = 10 - 2m - \frac{25}{2m}$, where m is the slope of the line which intersects $(2,5)$. Since $m \in (0,\infty)$ we find that at one of the endpoints,

$$\lim_{m \to 0^-} A(m) = +\infty,$$

so no absolute maximum exists.

44. We maximize the volume $V = \frac{1}{3}\pi r^2 h$, where $r^2 + h^2 = a^2$, so

$$V(h) = \frac{1}{3}\pi(a^2 - h^2)h, \qquad 0 \leq h \leq a.$$

Thus, $V'(h) = \frac{1}{3}\pi(a^2 - 3h^2) = 0 \implies h = \frac{a}{\sqrt{3}}$. By checking the endpoints, we see the maximum volume is $V(a/\sqrt{3}) = \frac{2}{27}\pi a^3 \sqrt{3}$.

5 We maximize the cross-sectional area

$$\begin{aligned} A(\theta) &= 10h + 2(\frac{1}{2}dh) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta) \\ &= 100(\sin\theta + \sin\theta\cos\theta), \quad 0 \leq \theta \leq \frac{\pi}{2}. \\ A'(\theta) &= 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(\cos\theta + 2\cos^2\theta - 1) = 100(2\cos\theta - 1)(\cos\theta + 1) \\ &= 0 \text{ when } \cos\theta = \frac{1}{2} \iff \theta = \frac{\pi}{3} \end{aligned}$$

since $\cos\theta \neq -1$ on $\theta \in [0, \frac{\pi}{2}]$. Now $A(0) = 0$, $A(\frac{\pi}{2}) = 100$ and $A(\frac{\pi}{3}) = 75\sqrt{3}$. Thus, the maximum occurs at $\theta = \frac{\pi}{3}$.

6 Let $(c, 1-c^2)$ be a point on the parabola, where $c \in (0,\infty)$. If $y(x) = 1-x^2$, then $y'(c) = -2c$, so the equation of the tangent line is $y-(1-c^2) = -2c(x-c)$. From this equation it is easy to show that the tangent line intersects the points $(0, c^2+1)$ and $(\frac{c^2+1}{2c}, 0)$. Therefore the area of the triangle is

$$A(c) = \frac{1}{2}\left(\frac{c^2+1}{2c}\right)(c^2+1) = \frac{1}{4}\frac{(c^2+1)^2}{c} = \frac{c^4+2c^2+1}{4c} = \frac{1}{4}\left(c^3+2c+\frac{1}{c}\right).$$

Solving $A'(c) = 0$ gives

$$\frac{1}{4}\left(3c^2+2-\frac{1}{c^2}\right) = 0 \implies 3c^2+2-\frac{1}{c^2} = 0 \implies 3c^4+2c^2-1 = 0$$
$$\implies (3c^2-1)(c^2+1) = 0 \implies c^2 = \frac{1}{3} \implies c = \frac{1}{\sqrt{3}}.$$

It is easy to see that at the endpoints $\lim_{c\to 0^+} A(c) = +\infty$ and $\lim_{c\to\infty} A(c) = +\infty$. Hence at the point $(\frac{1}{\sqrt{3}}, \frac{2}{3})$, the tangent line cuts from the first quadrant the triangle with smallest area.

7 (SHE 4.6)

10. $f(x) = x^3 - x^4 \implies f'(x) = 3x^2 - 4x^3 \implies f''(x) = 6x - 12x^2 = 6x(1-2x)$. Solving the inequality $6x(1-2x) > 0$, we find that f is concave down on $(-\infty, 0) \cup (\frac{1}{2}, \infty)$ and concave up on $(0, \frac{1}{2})$. Hence the points of inflection are $(0,0)$ and $(\frac{1}{2}, \frac{1}{16})$.

48. It is sufficient to show that the x-coordinate of the point of inflection is the x-coordinate of the midpoint of the line segment connecting the local extrema. It is easy to show that the x-coordinate of the point of inflection is $x_0 = -\frac{1}{3}a$. Now suppose that p has local extrema at x_1 and x_2, where $x_1 \neq x_2$. Then

$$p'(x_1) = p'(x_2) = 0 \implies 3x_1^2 + 2ax_1 + b - (3x_2^2 + 2ax_2 + b) = 0 \implies x_1 + x_2 = -\frac{2}{3}a.$$

Thus, $\frac{1}{2}(x_1+x_2) = -\frac{1}{3}a = x_0$.

8 (i) $\lim_{x\to\infty}\sqrt{x} = \infty$ means that for any $M > 0$, there exists $N > 0$ such that $x > M$ implies $\sqrt{x} > N$. Choose $N = \sqrt{M}$, then $x > M$ implies $\sqrt{x} > \sqrt{M} = N$, which is exactly what we needed to show.

(ii) $\lim_{x\to 2^-}\dfrac{1}{(x-2)^3} = -\infty$ means that for a sufficient large negative number $M < 0$, there exists $\delta > 0$ such that if $2-\delta < x < 2$, then $(x-2)^{-3} < M$.
Given $M < 0$, choose $\delta = \sqrt[3]{-1/M} > 0$. Then

$$2-\delta < x < 2 \implies -\delta < x-2 < 0 \implies -\frac{1}{\delta} > \frac{1}{x-2} \implies \frac{1}{(x-2)^3} < -\frac{1}{\delta^3} = M.$$

9 (i) $\displaystyle\lim_{x\to\infty}\frac{x^4+3x^2+1}{x^2(3x+1)(x-3)} = \lim_{x\to\infty}\frac{x^4+3x^2+1}{3x^4-8x^3-3x^2} = \lim_{x\to\infty}\frac{1+3x^{-2}+x^{-4}}{3-8x^{-1}-3x^{-2}} = \frac{1}{3}.$

(ii) Multiplying top and bottom by the conjugate,

$$\lim_{x\to-\infty}\frac{(x+\sqrt{x^2+5x})(x-\sqrt{x^2+5x})}{x-\sqrt{x^2+5x}} = \lim_{x\to-\infty}\frac{x^2-(x^2+5x)}{x-\sqrt{x^2+5x}} = \lim_{x\to-\infty}\frac{-5x}{x-\sqrt{x^2+5x}}$$
$$= \lim_{x\to-\infty}\frac{-5x}{x-\sqrt{x^2(1+\frac{5}{x})}} = \lim_{x\to-\infty}\frac{-5x}{x+x\sqrt{1+\frac{5}{x}}}$$

since as $x\to-\infty$, $\sqrt{x^2}=|x|=-x$. Hence,

$$\lim_{x\to-\infty}\frac{-5x}{x+x\sqrt{1+\frac{5}{x}}} = \lim_{x\to-\infty}\frac{-5}{1+\sqrt{1+\frac{5}{x}}} = -\frac{5}{2}.$$

10 (SHE 4.7)

8. Evaluating limits, we see that for $x\neq 0$, we have $\dfrac{\sqrt{x}}{4\sqrt{x}-x}=\dfrac{1}{4-\sqrt{x}}$, so

$$\lim_{x\to\infty}\frac{\sqrt{x}}{4\sqrt{x}-x}=0,\quad \lim_{x\to0^+}\frac{\sqrt{x}}{4\sqrt{x}-x}=\frac{1}{4},\quad \lim_{x\to16^+}\frac{\sqrt{x}}{4\sqrt{x}-x}=-\infty,\quad \lim_{x\to16^-}\frac{\sqrt{x}}{4\sqrt{x}-x}=\infty,$$

so $x=16$ is a vertical asymptote and $y=0$ is a horizontal asymptote.

22. For $y=3+x^{2/5}$ we have $y'=\frac{2}{5}x^{-3/5}$. Since $\lim_{x\to0^+}y'=+\infty$ and $\lim_{x\to0^-}y'=-\infty$, we see that at $x=0$ we have a cusp.

30. Differentiating gives us $f'(x)=(4x-3)(x-1)^{-2/3}$. Since $\lim_{x\to1^-}f'(x)=\lim_{x\to1^+}=\infty$, it follows we have a vertical tangent at $x=1$.

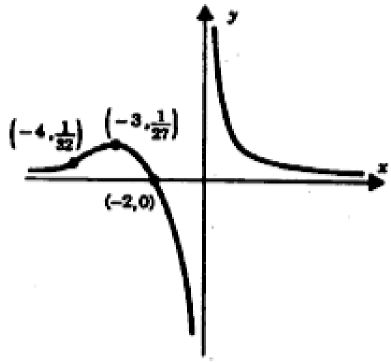
11 (i) (SHE 4.8 #12) We are given $f(x)=(x+2)/x^3=\frac{1}{x^2}+\frac{2}{x^3}$. The domain is $x\neq 0$. The lone intercept is $(-2,0)$. As for asymptotes, we evaluate the relevant limits:

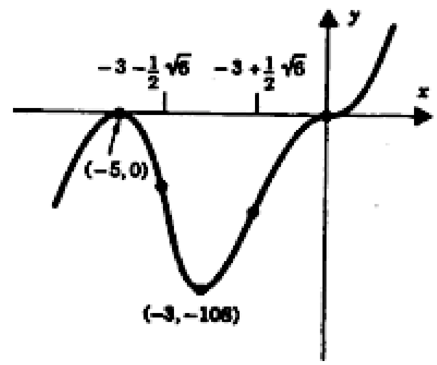
$$\lim_{x\to0^+}f(x)=+\infty,\quad \lim_{x\to0^-}f(x)=-\infty,\quad \lim_{x\to\pm\infty}f(x)=0,$$

hence $x=0$ is a vertical asymptote, and $y=0$ is a horizontal asymptote.

The first derivative is $f'(x)=-\frac{2}{x^3}-\frac{6}{x^4}=\frac{-2x-6}{x^4}$. The critical points are $x=-3$ and $x=0$, although the function is not defined at $x=0$. We now determine where f' is positive or negative and see by inspection that $f'(x)>0$ when $x<-3$ and $f'(x)<0$ when $x>-3$. Hence f is increasing on $x<-3$ and decreasing on $x>-3$. Hence $x=-3$ is a local maximum.

The second derivative is $f''(x)=\frac{6}{x^4}+\frac{24}{x^5}=\frac{6(x+4)}{x^5}$. We have $f''(x)=0$ when $x=-4$ and f'' is undefined when $x=0$ (where the function is undefined). Again, we solve the inequality $f''(x)>0$ to get $x\in(-\infty,-4)\cup(0,\infty)$; thus f is concave up on that interval, and f is concave down on $(-4,0)$. Therefore $x=-4$ and $x=0$ are inflection points (since concavity changes at both points). This yields the sketch below left.





(ii) (SHE 4.8 #36) We are given $f(x) = x^3(x+5)^2$. As the function is a polynomial, the domain is $\mathbb{R}$. The intercepts are $(0,0)$ and $(-5,0)$. As the function is a polynomial of degree 5, there are no vertical nor horizontal asymptotes.

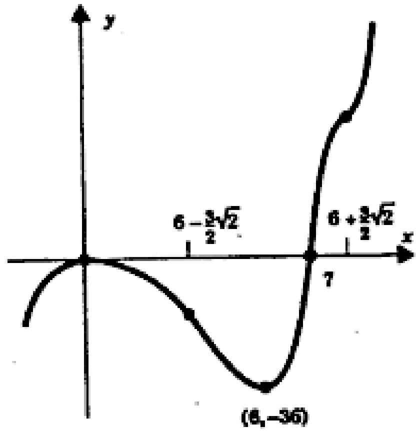
The first derivative is $f'(x) = 5x^2(x+3)(x+5)$. The critical points are $x = 0,-3,-5$. Solving $f'(x) > 0$ gives us $x \in (-\infty,-5)\cup(-3,0)\cup(0,\infty)$, so this is the interval for which f is increasing. Hence f is decreasing on $(-5,-3)$. Therefore the point $(-5,0)$ is a local maximum, $(-3,-108)$ is a local minimum, and $(0,0)$ is not a local extrema.

Evaluating the second derivative, we get $f''(x) = 10x(2x^2+12x+15)$. The candidates for inflection points (by applying the quadratic formula) are $x = 0,-3-\frac{1}{2}\sqrt{6},-3+\frac{1}{2}\sqrt{6}$. Solving $f''(x) > 0$ gives us $(-3-\frac{1}{2}\sqrt{6},-3+\frac{1}{2}\sqrt{6})\cup(0,\infty)$, so this is the interval for which f is concave up. Thus, f is concave down on $(-\infty,-3-\frac{1}{2}\sqrt{6})\cup(-3+\frac{1}{2}\sqrt{6},0)$. Therefore all candidates of inflection points are indeed inflection points. This yields the sketch above right.

(iii) (SHE 4.8 #42) We are given $f(x) = x^2(x-7)^{1/3}$. The domain is $\mathbb{R}$ and the intercepts are $(0,0)$ and $(7,0)$. There are no vertical or horizontal asymptotes since $\lim_{x\to\pm\infty} f(x)$ does not exist and the domain is $\mathbb{R}$.

The first derivative is $f'(x) = \dfrac{7x(x-6)}{3(x-7)^{2/3}}$. The critical points are $x = 0,6,7$. By evaluating where $f'(x) > 0$ and $f'(x) < 0$, we see that f is increasing on $(-\infty,0)\cup(6,\infty)$ and decreasing on $(0,6)$. Hence $(0,0)$ is a local maximum and $(6,-36)$ is a local minimum.

The second derivative is $f''(x) = \dfrac{14(2x^2-24x+63)}{9(x-7)^{5/3}}$. We find the candidates for inflection points are $x = 6-\frac{3}{2}\sqrt{2}$, $x = 6+\frac{3}{2}\sqrt{2}$, and $x = 7$. The function f is concave down on $(-\infty,6-\frac{3}{2}\sqrt{2})\cup(7,6+\frac{3}{2}\sqrt{2})$ and concave up on $(6-\frac{3}{2}\sqrt{2})\cup(6+\frac{3}{2}\sqrt{2},\infty)$. Hence all candidates are indeed inflection points. This yields the sketch below left.



12 (SHE 4.12)

6. The recursion formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. For $f(x) = \sin x - x^2$, we have $f'(x) = \cos x - 2x$. Hence, $x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2}{\cos x_n - 2x_n} = \dfrac{x_n \sin x_n - 2x_n^2 - \sin x_n + x_n^2}{\cos x_n - 2x_n} = \dfrac{x_n \sin x_n - x_n^2 - \sin x_n}{\cos x_n - 2x_n}$. With a calculator, we find that $x_4 \approx 0.87673$.

14(a) Let $f(x) = x^k - a$. Then $f'(x) = kx^{k-1}$. The Newton-Raphson method applied to this function gives

$$x_{n+1} = x_n - \frac{x_n^k - a}{kx_n^{k-1}} = x_n - \frac{1}{k}x_n + \frac{1}{k}\frac{a}{x_n^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{a}{x_n^{k-1}}\right].$$

13 (i) $\displaystyle\lim_{x\to 0} \frac{1-\cos x}{x^2+x} \overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{\sin x}{2x+1} = 0.$

(ii) Applying L'Hôpital's Rule twice we get

$$\lim_{x\to 0} \frac{\tan x - x - \frac{1}{3}x^3}{x^5} \overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{\sec^2 x - 1 - x^2}{5x^4} \overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{2\sec^2 x \tan x - 2x}{20x^3} = \lim_{x\to 0} \frac{\sec^2 x \tan x - x}{10x^3}$$
$$= \lim_{x\to 0} \frac{\tan^3 x + \tan x - x}{10x^3}.$$

Applying L'Hôpital's Rule again gives

$$\lim_{x\to 0} \frac{3\tan^2 x \sec^2 x + \sec^2 x - 1}{30x^2} = \lim_{x\to 0} \frac{3\tan^4 x + 3\tan^2 x + \tan^2 x}{30x^2} = \lim_{x\to 0} \frac{3\tan^4 x + 4\tan^2 x}{30x^2}$$
$$= \lim_{x\to 0} \frac{\sin^4 x}{10x^2\cos^4 x} + \frac{2\sin^2 x}{15x^2\cos^2 x} = 0 + \frac{2}{15} = \frac{2}{15}.$$

(iii) $\displaystyle\lim_{x\to 0} \frac{4}{x^2} - \frac{2}{1-\cos x} = \lim_{x\to 0} \frac{4 - 4\cos x - 2x^2}{x^2(1-\cos x)} \overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{4\sin x - 4x}{2x - 2x\cos x + x^2 \sin x}.$

Applying L'Hôpital's Rule three more times,

$$\lim_{x\to 0} \frac{4\cos x - 4}{2 - 2\cos x + 4x\sin x + x^2\cos x} \overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{-4\sin x}{6\sin x + 6x\cos x - x^2\sin x}$$
$$\overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{-4\cos x}{12\cos x - 8x\sin x - x^2\cos x} = -\frac{1}{3}.$$

14 We have $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0,\\ 1, & x=0.\end{cases}$

(a) $f'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\dfrac{f(h)-1}{h}=\lim_{h\to 0}\dfrac{\frac{\sin h}{h}-1}{h}=\lim_{h\to 0}\dfrac{\sin h-h}{h^2}\overset{\mathrm{H}}{=}\lim_{h\to 0}\dfrac{\cos h-1}{2h}\overset{\mathrm{H}}{=}$ $\lim_{h\to 0}\dfrac{-\sin h}{2}=0.$

(b) Differentiating for $x\neq 0$, we get $f'(x)=\begin{cases}\frac{x\cos x-\sin x}{x^2}, & x\neq 0,\\ 0, & x=0.\end{cases}$

Evaluating, we get

$$\begin{aligned}f''(0)&=\lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0}\frac{f'(h)}{h}=\lim_{h\to 0}\frac{h\cos h-\sin h}{h^3}\\ &\overset{\mathrm{H}}{=}\lim_{h\to 0}\frac{\cos h-h\sin h-\cos h}{3h^2}=\lim_{h\to 0}\frac{-\sin h}{3h}=-\frac{1}{3}.\end{aligned}$$