MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 10

1 (SHE 7.4)

16. $y = x^2e^x - xe^{x^2} \implies y' = 2xe^x + x^2e^x - e^{x^2} - 2x^2e^{x^2}$.
30. $\displaystyle\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2} + C$.
40. $\displaystyle\int \frac{\sin(e^{-2x})}{e^{2x}}\,dx = \int e^{-2x}\sin(e^{-2x})\,dx = \frac{1}{2}\cos(e^{-2x}) + C$.
56. If $y = e^x$ then $x = \ln y$. The area of the rectangle is $A = y\ln y$. Therefore
$$\frac{dA}{dt} = (1+\ln y)\frac{dy}{dt}.$$
At $y = 3$, we have $\frac{dy}{dt} = \frac{1}{2}$, so $\frac{dA}{dt} = \frac{1}{2}(1+\ln 3)$ square units per minute.

2. (SHE 7.5)

24. $h(x) = 7^{\sin x^2} \implies h'(x) = 7^{\sin x^2}(\ln 7)(\cos x^2)2x$.
36. Notice that $c = b^{\log_b c}$. Then $\log_a c = \log_a(b^{\log_b c}) = (\log_b c)(\log_a b)$.
50. Applying logarithmic differentiation, $y = x^{x^2} \implies \ln y = x^2\ln x$.
Therefore $\frac{y'}{y} = x + 2x\ln x \implies y' = x^{x^2}(x + 2x\ln x)$.
60. $\displaystyle\int_0^1 4^x\,dx = \left[\frac{4^x}{\ln 4}\right]_0^1 = \frac{3}{\ln 4}$.

3. (SHE 7.6)

36. Suppose $f'(t) = (\sin t)f(t)$. Then $f'(t) - (\sin t)f(t) = 0$. Multiplying both sides by $e^{\cos t}$,
$$e^{\cos t}f'(t) - (\sin t)e^{\cos t}f(t) = 0 \implies \frac{d}{dt}\left[e^{\cos t}f(t)\right] = 0 \implies e^{\cos t}f(t) = C.$$
Hence $f(t) = Ce^{-\cos t}$.
38. Clearly $f(t) = 0$ is a solution to the equation $f'(t) = g(t)f(t)$. Otherwise, rewrite the equation as $f'(t) - g(t)f(t) = 0$ and set $h(t) = -\int g(t)\,dt$. Then
$$e^{h(t)}f'(t) - g(t)e^{h(t)}f(t) = 0 \implies \left[e^{h(t)}f(t)\right]' = 0 \implies e^{h(t)}f(t) = C.$$
Therefore $f(t) = Ce^{-h(t)} = Ce^{\int g(t)\,dt}$.

4. (SHE 7.7)

16. $f(x) = e^{\arctan x} \implies f'(x) = e^{\arctan x}\frac{1}{1+x^2}$.
30. $f(x) = e^{\sec^{-1}x} \implies f'(x) = e^{\sec^{-1}x}\cdot\frac{1}{|x|\sqrt{x^2-1}}$.
54. Let $u = \tan x$, then $du = \sec^2 x\,dx$. Then $\displaystyle\int \frac{\sec^2 x}{\sqrt{9-\tan^2 x}}\,dx = \int \frac{du}{\sqrt{9-u^2}} = \arcsin\frac{u}{3} + C =$ $\displaystyle\arcsin\frac{\tan x}{3} + C$.

58. Let $u = \sin x \implies du = \cos x\, dx$.
$$\int \frac{\cos x}{3+\sin^2 x}\, dx = \int \frac{du}{3+u^2} = \frac{1}{\sqrt{3}} \arctan \frac{u}{\sqrt{3}} + C = \frac{1}{\sqrt{3}} \arctan \frac{\sin x}{\sqrt{3}} + C.$$

5. Consider the equation $\ln x = cx^2$. It is clear via a diagram that when $c \leq 0$ the equation has exactly one solution. If $c > 0$, there is a unique curve $y = cx^2$ which intersects $y = \ln x$ once; at the point a, they share the same tangent line. Therefore
$$\ln a = ca^2, \text{ and } \qquad \frac{1}{a} = 2ca \implies a^2 = \frac{1}{2c}.$$
Therefore
$$\ln a = ca^2 = c \cdot \frac{1}{2c} = \frac{1}{2} \implies a = e^{1/2} \implies c = \frac{\ln a}{a^2} = \frac{1}{2e}.$$
Therefore $c \leq 0$ or $c = \frac{1}{2e}$.

6. (i) By properties of exponents, $\lim_{x\to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x}$. Since (by letting $x = \frac{1}{t}$),
$$\lim_{x\to 0^+} x \ln x = \lim_{t\to\infty} \frac{\ln(1/t)}{t} = \lim_{t\to\infty} \frac{-\ln t}{t} \overset{\text{H}}{=} 0,$$
so it follows that $\lim_{x\to 0^+} x^x = e^0 = 1$.

(ii) By continuity of e^x, we have $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\ln(1+\frac{a}{x})} = e^{\lim_{x\to\infty} x \ln\left(1+\frac{a}{x}\right)}$. Now
$$\lim_{x\to\infty} x\ln\left(1+\frac{a}{x}\right) = \lim_{t\to 0^+} \frac{\ln(1+at)}{t} \overset{\text{H}}{=} \lim_{t\to 0^+} \frac{1}{1+at} \cdot a = a \implies \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a.$$
In particular, note that for the case $a = 1$, $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

7. If $f(x)$ is continuous, then by Theorem 5.3.5 (First Fundamental Theorem of Calculus) we differentiate both sides to get $f'(x) = f(x)$. By Theorem 7.6.1 it follows that $f(x) = Ce^x$ for some constant C. But note that $f(0) = 0$, so it follows that $C = 0$. Therefore the only function which satisfies $\int_0^x f(t)\, dt = f(x)$ is identically $f(x) = 0$.

9. (a) Consider the graph of $y = e^{x^2}$, which is positive and increasing for all x. It follows that the area under the curve between x and $x + \frac{\ln x}{2x}$ is larger than the rectangle with vertices located at $(x, 0)$, $(x + \frac{\ln x}{2x}, 0)$ and (x, e^{x^2}). That is,
$$\int_x^{x+\frac{\ln x}{2x}} e^{t^2}\, dt > e^{x^2} \cdot \frac{\ln x}{2x} = \frac{e^{x^2} \ln x}{2x}.$$
(Alternatively, consider the partition $P = \{x, x + \frac{\ln x}{2x}\}$ and consider $L_f(P) < \int_x^{x+\frac{\ln x}{2x}} e^{t^2}\, dt$.)

(b) Note that by L'Hôpital's Rule
$$\lim_{x\to\infty} \frac{e^x \ln x}{2x} = \lim_{x\to\infty} \ln x \cdot \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{H}}{=} \lim_{x\to\infty} \ln x \cdot \lim_{x\to\infty} \frac{e^x}{2} = \infty,$$

so it follows that $\lim_{x\to\infty}\int_x^{x+\frac{\ln x}{2x}} e^{t^2}\ dt = \infty$, so by L'Hôpital's Rule

$$\begin{aligned}
\lim_{x\to\infty}\frac{\int_x^{x+\frac{\ln x}{2x}} e^{t^2}\ dt}{e^{x^2}} &= \lim_{x\to\infty}\frac{\int_0^{x+\frac{\ln x}{2x}} e^{t^2}\ dt - \int_0^x e^{t^2}\ dt}{e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^{(x+\frac{\ln x}{2x})^2}\left[1+\frac{2(1-\ln x)}{4x^2}\right]-e^{x^2}}{2xe^{x^2}} \\
&= \lim_{x\to\infty}\frac{e^{x^2+\ln x+\frac{(\ln x)^2}{4x^2}}\left[1+\frac{(1-\ln x)}{2x^2}\right]-e^{x^2}}{2xe^{x^2}} = \lim_{x\to\infty}\frac{xe^{(\frac{\ln x}{x})^2}e^{x^2}\left[1+\frac{(1-\ln x)}{2x^2}\right]-e^{x^2}}{2xe^{x^2}} \\
&= \lim_{x\to\infty}\frac{xe^{(\frac{\ln x}{x})^2}\left[1+\frac{(1-\ln x)}{2x^2}\right]-e^{x^2}}{2x} \\
&= \lim_{x\to\infty}\frac{1}{2}e^{(\frac{\ln x}{2x})^2}\left[1+\frac{1-\ln x}{2x^2}\right] = \frac{1}{2},
\end{aligned}$$

since $\lim_{x\to\infty}\frac{\ln x}{2x} = 0$ and $\lim_{x\to\infty}\frac{1-\ln x}{x^2} = 0$, both which are easily solved by L'Hôpital's Rule.