마이크로파 microwave μ-wave

$v = f\lambda$

우선통신에 사용되는 전자파의 파장이 비교적 짧은 wave

1 ~ 30 GHz

(파장 30cm ~ 0.1 cm)

↓

실제로는 좀더 넓은 범위 1 ~ 3000 GHz.

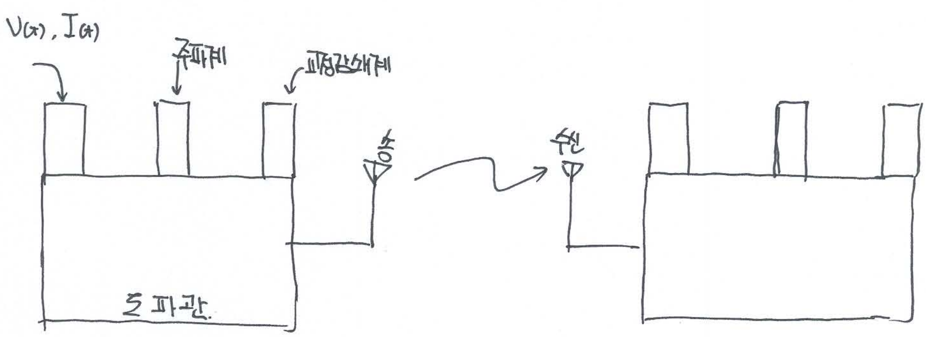
(파장λ 30cm - 0.1 mm)

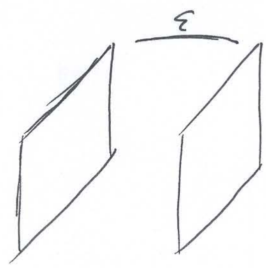
파장

분포정수형.

(송신측) $\xrightarrow[\text{극전선}]{\text{도파관. 동축.}}$ (수신측)

(μwave 발생)

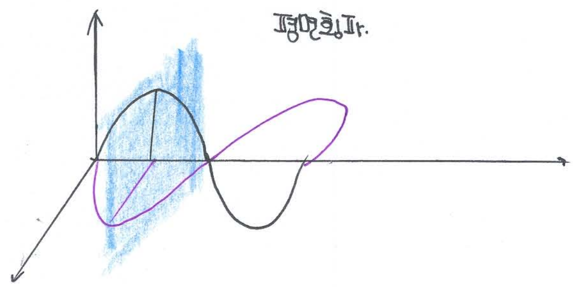




R
L
C

Guide - 전송선로.

TEM (transmission Electromagnetic)



Chp3. 전송선로.

{ 어떤 한 곳에서 다른 곳으로 에너지를 전송하거나 유도 (Guide) 하기 위한 장치 }

① parallel - plate transmission Line
평행판 전송 선로

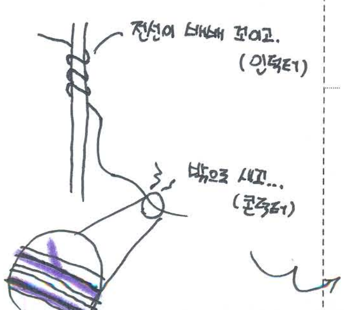
② two - wire

③ Coaxial 동축

3.1 전송 선로 방정식

전송선로에 전류를 인가하면,

① 도선의 고유저항 발생 (전압강하)
② 자체 성분에 의한 인덕턴스 (위상차 발생)
   - 전류 위상 지연을 일으킴.
③ 누설저항 (누설 콘덕턴스)
④ 전체에 의한 정전 용량.



"저항, 인덕턴스" 는 직렬형태

"콘덕턴스. 정전용량" 은 병렬 형태

매질의 특성값

$$\begin{cases} \varepsilon \\ \mu \\ \sigma \end{cases}$$

식(3) * cheng 381p.

↓

$\gamma = \alpha + j\beta$

$= \sqrt{(R+j\omega L)(G+j\omega C)}\ m^{-1}$

* Cheng 책 382page 참고.

* 전송선로 파라미터 parameter

RLGC 와 전송선로 구조에서

*식(3) 에서 R을 소거 (도체 고유저항 小) cheng382p.

$$\gamma = j\omega\sqrt{LC}\left(1 + G/j\omega C\right)^{1/2} \quad -(4)$$

TEM에서 보면, 즉 손실매질 "ε, μ, σ"

$$\gamma = j\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2} \quad -(5)$$

$L \rightarrow \mu$ (투자율)

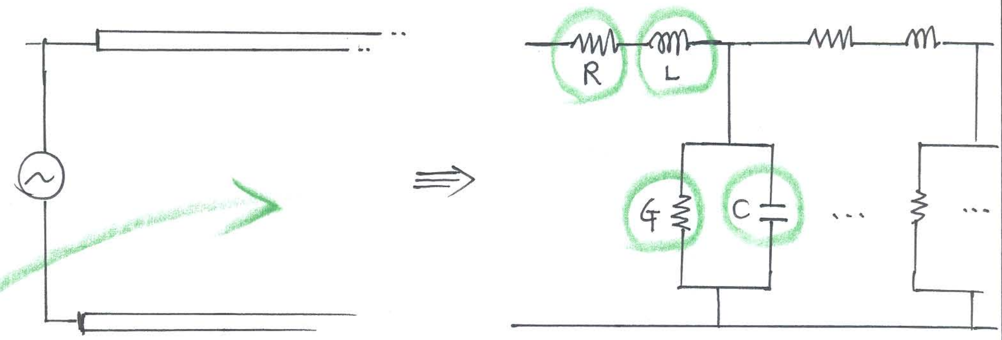
$C \rightarrow \varepsilon$

$G \rightarrow \sigma$ (도전율)

$$\boxed{\frac{G}{C} = \sigma/\varepsilon, \quad LC = \mu\varepsilon}$$

전송선로와 매질변수와의 관계.

※ 61p. 그림. 3.4.



$R\ [\Omega/m] \leftrightarrow G\ [\mho/m,\ S/m]$

지멘스.

$L\ [H/m]$

$C\ [F/m]$

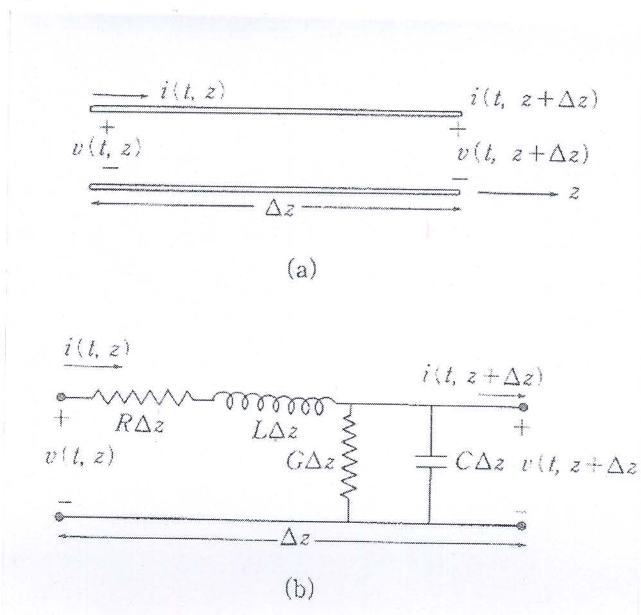
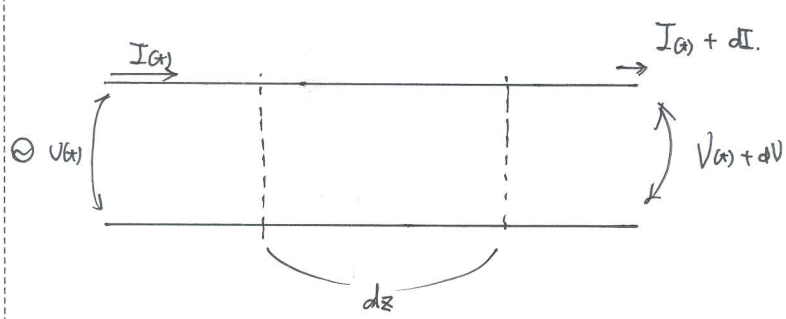
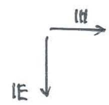
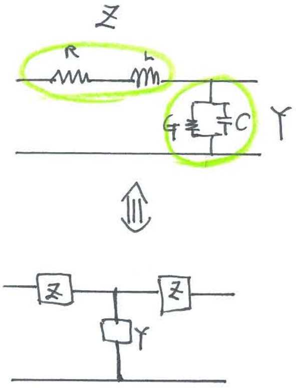


그림 3.4 미소한 구간의 전송선로의 등가회로.







$$\begin{cases} Z = R + X_L \\ Y = G + 1/X_c \end{cases}$$

- 미소 길이 dz 에 의한 전압 강하 (−)

$$dV(z) = (-)\,I(z) \cdot Z \cdot dz$$

where.

$$Z = R + X_L = R + j\omega L$$

= Impedance unit length.

- 미소길이 dz 에 의한 전류강하.

$$dI(z) = -V(z) \cdot Y \cdot dz$$

where.

$$Y = G + 1/X_c = G + j\omega C$$

= Admitance Unit length

(z 방향 →)

$$\therefore \quad \frac{dV(z)}{dz} = -z\, I(z)$$

$$\frac{dI(z)}{dz} = -Y\, V(z)$$

(1)식. 전송선로에서 때는 전압 전류 방정식.

식 (1)을 다시 미분하여 정리를 하면.

$$\frac{d^2V(z)}{dz^2} = -z\frac{dI(z)}{dz} = -z\cdot Y\cdot V(z) = -\gamma^2\, V(z)$$

$$\frac{d^2I(z)}{dz^2} = -Y\frac{dV(z)}{dz} = -z\cdot Y\cdot I(z) = -\gamma^2\, I(z)$$

(2)

※ (2) 헬름홀츠 방정식. where. $\gamma = \sqrt{z\cdot Y}$

※ $\nabla^2 E + K^2 E = 0$

- 헬름홀츠.*

google. wiki 검색!*

(3)

$$= \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{\underbrace{(RG-\omega^2 LC)}_{Real} + \underbrace{j\omega\,(LG+RGC)}_{imagina.\ 허수}}$$

= Complex 형태.

$= \alpha + j\beta$ (α = 감쇠상수 [Np/m], β = 위상정수 [rad/m])

= propagation Constant.

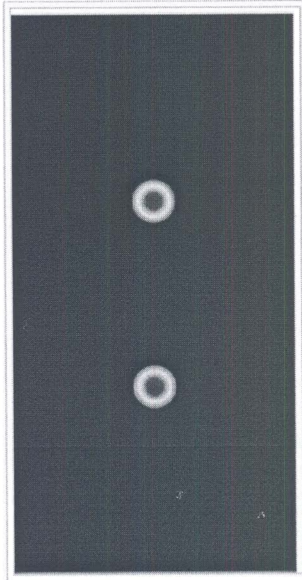
Helmholtz equation

From Wikipedia, the free encyclopedia

The **Helmholtz equation**, named for Hermann von Helmholtz, is the elliptic partial differential equation

$$(\nabla^2 + k^2)A = 0$$

where ∇^2 is the Laplacian, k is a constant, and the unknown function $A = A(x,y,z)$ is defined on n-dimensional Euclidean space $\mathbf{R}^n$ (typically $n = 1, 2$, or 3, when the solution to this equation makes physical sense).

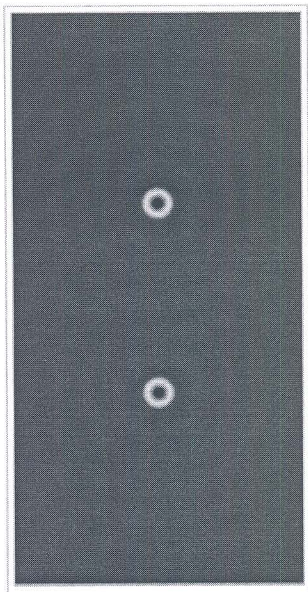


Two sources of radiation in the plane, given mathematically by a function f which is zero in the blue region.

Contents

Motivation and uses

The Helmholtz equation often arises in the study of physical problems involving partial differential equations (PDEs) in both space and time. The Helmholtz equation, which represents the **time-independent** form of the original equation, results from applying the technique of separation of variables to reduce the complexity of the analysis.



The real part of the resulting field A, A is the solution to the inhomogeneous Helmholtz equation $(\nabla^2 + k^2)A = -f.$

For example, consider the wave equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\mathbf{r}, t) = 0.$$

Separation of variables begins by assuming that the wave function $u(\mathbf{r}, t)$ is in fact separable:

$$u(\mathbf{r}, t) = A(\mathbf{r}) \cdot T(t)$$

Substituting this form into the wave equation, and then simplifying, we obtain the following equation:

$$\frac{\nabla^2 A}{A} = \frac{1}{c^2 T}\frac{d^2 T}{dt^2}$$

Notice the expression on the left-hand side depends only on $\mathbf{r}$, whereas the right-hand expression depends only on t. As a result, this equation is valid in the general case if and only if both sides of the equation are equal to a constant value. From this observation, we obtain two equations, one for $A(\mathbf{r})$, the other for $T(t)$:

$$\frac{\nabla^2 A}{A} = -k^2$$

and

$$\frac{1}{c^2 T}\frac{d^2 T}{dt^2} = -k^2$$

where we have chosen, without loss of generality, the expression $-k^2$ for the value of the constant.

Rearranging the first equation, we obtain the Helmholtz equation:

$$\nabla^2 A + k^2 A = (\nabla^2 + k^2) A = 0.$$

Likewise, after making the substitution

$$\omega \stackrel{\text{def}}{=} kc$$

the second equation becomes

$$\frac{d^2 T}{dt^2} + \omega^2 T = \left(\frac{d^2}{dt^2} + \omega^2 \right) T = 0,$$

where k is the wave vector and ω is the angular frequency.

Harmonic solutions

It is relatively easy to show that solutions to the Helmholtz equation will take the form:

$$A(\mathbf{r}) = C_1 e^{i\mathbf{k}\cdot\mathbf{r}} + C_2 e^{-i\mathbf{k}\cdot\mathbf{r}}.$$

which corresponds to the time-harmonic solution

$$T(t) = D_1 e^{i\omega t} + D_2 e^{-i\omega t}$$

for arbitrary (complex-valued) constants C and D, which will depend on the initial conditions and boundary conditions, and subject to the dispersion relation:

$$k = |\mathbf{k}| = \frac{\omega}{c}$$

We now have Helmholtz's equation for the spatial variable $\mathbf{r}$ and a second-order ordinary differential equation in time. The solution in time will be a linear combination of sine and cosine functions, with angular frequency of ω, while the form of the solution in space will depend on the boundary conditions. Alternatively, integral transforms, such as the Laplace or Fourier transform, are often used to transform a hyperbolic PDE into a form of the Helmholtz equation.

Because of its relationship to the wave equation, the Helmholtz equation arises in problems in such areas of physics as the study of electromagnetic radiation, seismology, and acoustics.

Solving the Helmholtz equation using separation of variables

The general solution to the spatial Helmholtz equation

$$(\nabla^2 + k^2) A = 0$$

can be obtained using separation of variables.

Vibrating membrane

The two-dimensional analogue of the vibrating string is the vibrating membrane, with the edges clamped to be motionless. The Helmholtz equation was solved for many basic shapes in the 19th century: the rectangular membrane by Siméon Denis Poisson in 1829, the equilateral triangle by Gabriel Lamé in 1852, and the circular membrane by Alfred Clebsch in 1862. The elliptical drumhead was studied by Emile Mathieu, leading to Mathieu's differential equation. The solvable shapes all correspond to shapes whose dynamical billiard table is integrable, that is, not chaotic. When the motion on a correspondingly-shaped billiard table is chaotic, then no closed form solutions to the Helmholtz equation are known. The study of such systems is known as quantum chaos, as the Helmholtz equation and similar equations occur in quantum mechanics.

If the edges of a shape are straight line segments, then a solution is integrable or knowable in closed-form only if it is expressible as a finite linear combination of plane waves that satisfy the boundary conditions (zero at the boundary, i.e., membrane clamped).

An interesting situation happens with a shape where about half of the solutions are integrable, but the remainder are not. A simple shape where this happens is with the regular hexagon. If the wavepacket describing a quantum billiard ball is made up of only the closed-form solutions, its motion will not be chaotic, but if any amount of non-closed-form solutions are included, the quantum billiard motion becomes chaotic. Another simple shape where this happens is with an "L" shape made by reflecting a square down, then to the right.

If the domain is a circle of radius a, then it is appropriate to introduce polar coordinates r and θ. The Helmholtz equation takes the form

$$A_{rr} + \frac{1}{r}A_r + \frac{1}{r^2}A_{\theta\theta} + k^2 A = 0.$$

We may impose the boundary condition that A vanish if $r=a$; thus

$$A(a,\theta) = 0.$$

The method of separation of variables leads to trial solutions of the form

$$A(r,\theta) = R(r)\Theta(\theta),$$

where Θ must be periodic of period 2π. This leads to

$$\Theta'' + n^2\Theta = 0,$$

and

$$r^2R'' + rR' + r^2k^2R - n^2R = 0.$$

It follows from the periodicity condition that

$$\Theta = \alpha\cos n\theta + \beta\sin n\theta,$$

and that n must be an integer. The radial component R has the form

$$R(r) = \gamma J_n(\rho),$$

where the Bessel function $J_n(\rho)$ satisfies Bessel's equation

$$\rho^2 J_n'' + \rho J_n' + (\rho^2 - n^2)J_n = 0,$$

and $\rho=kr$. The radial function J_n has infinitely many roots for each value of n, denoted by $\rho_{m,n}$. The boundary condition that A vanishes where $r=a$ will be satisfied if the corresponding frequencies are given by

$$k_{m,n} = \frac{1}{a}\rho_{m,n}.$$

The general solution A then takes the form of a doubly infinite sum of terms involving products of

$$\sin(n\theta) \text{ or } \cos(n\theta), \text{ and } J_n(k_{m,n}r).$$

These solutions are the modes of vibration of a circular drumhead.

Three-dimensional solutions

In spherical coordinates, the solution is:

$$A(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_{nlm}j_n(kr) + b_{nlm}y_n(kr))Y_l^m(\theta,\phi).$$

This solution arises from the spatial solution of the wave equation and diffusion equation. Here $j_n(kr)$ and $y_n(kr)$ are the spherical Bessel functions, and

$$Y_l^m(\theta,\phi)$$

are the spherical harmonics (Abramowitz and Stegun, 1964). Note that these forms are general solutions, and require boundary conditions to be specified to be used in any specific case. For infinite exterior domains, a radiation condition may also be required (Sommerfeld, 1949).

For $a = (x,y,z)$ function $A(a)$ has asymptotics

$$A(a) = \frac{e^{ik|a|}}{|a|} f(a/|a|, k, u_0) + o(1/|a|) \text{ when } a \to \infty$$

where function f is called scattering amplitude and $u_0(a)$ is the value of A at each boundary point a.

Paraxial form

The paraxial form of the Helmholtz equation is:

$$\nabla_T^2 A - j2k \frac{\partial A}{\partial z} = 0$$

where

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the transverse form of the Laplacian.

This equation has important applications in the science of optics, where it provides solutions that describe the propagation of electromagnetic waves (light) in the form of either paraboloidal waves or Gaussian beams. Most lasers emit beams that take this form.

In the paraxial approximation, the complex magnitude of the electric field E becomes

$$E(\mathbf{r}) = A(\mathbf{r}) e^{-jkz}$$

where A represents the complex-valued amplitude of the electric field, which modulates the sinusoidal plane wave represented by the exponential factor.

The paraxial approximation places certain upper limits on the variation of the amplitude function A with respect to longitudinal distance z. Specifically:

$$\left|\frac{\partial A}{\partial z}\right| \ll |kA|$$

and

$$\left|\frac{\partial^2 A}{\partial z^2}\right| \ll |k^2 A|$$

These conditions are equivalent to saying that the angle θ between the wave vector $\mathbf{k}$ and the optical axis z must be small enough so that

$$\sin(\theta) \approx \theta \qquad \text{and} \qquad \tan(\theta) \approx \theta$$

The paraxial form of the Helmholtz equation is found by substituting the above-stated complex magnitude of the electric field into the general form of the Helmholtz equation as follows.

$$\nabla^2 (A(x,y,z)\, e^{-jkz}) + k^2 (A(x,y,z)\, e^{-jkz}) = 0$$

Expansion and cancellation yields the following:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(A\left(x,y,z\right)e^{-jkz}\right)+\left(\frac{\partial^2}{\partial z^2}A\left(x,y,z\right)\right)e^{-jkz}-2\left(\frac{\partial}{\partial z}A\left(x,y,z\right)\right)jke^{-jkz}=0.$$

Because of the paraxial inequalities stated above, the $\partial^2A/\partial z^2$ factor is neglected in comparison with the $\partial A/\partial z$ factor. The yields the Paraxial Helmholtz equation.

Inhomogeneous Helmholtz equation

The **inhomogeneous Helmholtz equation** is the equation

$$\nabla^2 A + k^2 A = -f \text{ in } \mathbb{R}^n$$

where $f:\mathbb{R}^n \to \mathbb{C}$ is a given function with compact support, and $n = 1,2,3$.

In order to solve this equation uniquely, one needs to specify a boundary condition at infinity, which is typically the Sommerfeld radiation condition

$$\lim_{r\to\infty} r^{\frac{n-1}{2}}\left(\frac{\partial}{\partial r}-ik\right)A(r\hat{x})=0$$

uniformly in $\hat{x}$ with $|\hat{x}|=1$, where the vertical bars denote the Euclidean norm.

With this condition, the solution to the inhomogeneous Helmholtz equation is the convolution

$$A(x)=(G*f)(x)=\int_{\mathbb{R}^n} G(x-y)f(y)\,dy$$

(notice this integral is actually over a finite region, since *f* has compact support). Here, *G* is the Green's function of this equation, that is, the solution to the inhomogeneous Helmholtz equation with *f* equaling the Dirac delta function, so *G* satisfies

$$\nabla^2 G + k^2 G = -\delta \text{ in } \mathbb{R}^n.$$

The expression for the Green's function depends on the dimension of the space. One has

$$G(x)=\frac{ie^{ik|x|}}{2k}$$

for $n = 1$,

$$G(x)=\frac{i}{4}H_0^{(1)}(k|x|)$$

for $n = 2$, where $H_0^{(1)}$ is a Hankel function, and

$$G(x)=\frac{e^{ik|x|}}{4\pi|x|}$$

for $n = 3$.

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External links

- Helmholtz Equation at EqWorld: The World of Mathematical Equations.
- Vibrating Circular Membrane by Sam Blake, The Wolfram Demonstrations Project.

Categories: Waves | Elliptic partial differential equations