Lemma 1. Suppose $E \subset \mathbb{R}^n$ be a measurable set and f, g be nonnegative integrable functions on E. Let's define $\omega_f(t) = m\{x \in E \,|\, f(x) > t\}$ and ω_g likewise. Also, suppose there is t_0 such that $\omega_f - \omega_g \le 0$ on $(-\infty, t_0)$ and $\omega_f - \omega_g \ge 0$ on (t_0, ∞). If $\int_E f^{p_0} \ge \int_E g^{p_0}$ for some $p_0 > 0$, then $\int_E f^p \ge \int_E g^p$ for all $p > p_0$ satisfying $f^p, g^p \in L(E)$. Strict inequality holds if $|\omega_f - \omega_g| \ne 0$ on a set with positive measure.

proof. It is clear if $t_0 \le 0$. So we may assume $t_0 > 0$. From the general measure theory, we know that

$$\int_E f^p = p \int_0^\infty t^{p-1} \omega_f(t)\, dt.$$

Then

$$\begin{aligned}
\int_E f^p - \int_E g^p &= p \int_0^\infty t^{p-1} (\omega_f(t) - \omega_g(t))\, dt \\
&= p \int_0^{t_0} t^{p-1} (\omega_f(t) - \omega_g(t))\, dt + p \int_{t_0}^\infty t^{p-p_0} t^{p_0-1} (\omega_f(t) - \omega_g(t))\, dt \\
&\ge p \int_0^{t_0} t^{p-1} (\omega_f(t) - \omega_g(t))\, dt + p \int_{t_0}^\infty t_0^{p-p_0} t^{p_0-1} (\omega_f(t) - \omega_g(t))\, dt \qquad (1) \\
&\ge p \int_0^{t_0} t^{p-1} (\omega_f(t) - \omega_g(t))\, dt - p \int_0^{t_0} t_0^{p-p_0} t^{p_0-1} (\omega_f(t) - \omega_g(t))\, dt \\
&= p \int_0^{t_0} (t^{p-p_0} - t_0^{p-p_0}) t^{p_0-1} (\omega_f(t) - \omega_g(t))\, dt \\
&\ge 0. \qquad (2)
\end{aligned}$$

Inequality (1) or (2) can be modified to be strict if $|\omega_f - \omega_g| \ne 0$ on a set with positive measure. □

From now on, we fix $E = [0, \infty)$ and

$$f(x) = e^{-x^2/2\pi}, \qquad g(x) = \left| \frac{\sin x}{x} \right|.$$

Lemma 2. $\int_0^\infty f^p = \pi/\sqrt{2p}$ and for $0 < x < \pi$, $g(x) < f(x)$.

proof. The first equality is an easy consequence of the Gaussian integral. To show the inequality, note that $e^x > 1 + x$ for any nonzero real x. Then for $0 < x < \pi$,

$$g(x) = \prod_{n=1}^\infty \left(1 - \frac{x^2}{n^2\pi^2}\right) < \prod_{n=1}^\infty e^{-x^2/(n\pi)^2} = e^{-x^2/6} < f(x).$$

□

Now we are ready to prove the main theorem.

Theorem. For $p \ge 2$,

$$\int_0^\infty \left| \frac{\sin x}{x} \right|^p dx \le \frac{\pi}{\sqrt{2p}}$$

with equality holds if and only if $p = 2$.

proof. It is easy to see that equality holds if $p = 2$. By noting that the right hand side of the given inequality coincides the integral of f^p on $(0, \infty)$, the inequality that we have to prove recudes to $\int_0^\infty f^p > \int_0^\infty g^p$ for $p > 2$. For this, it is enough to show that $\omega_f - \omega_g$ satisfies the conditions given in Lemma 1.

First, let's determine ω_f and ω_g in explicit form. It is easy to see that $\omega_f(t) = f^{-1}(t) = \sqrt{2\pi \log(1/t)}$ on $(0, 1)$ and $\omega_f(t) = \omega_g(t) = 0$ for $t \geq 1$. An explicit form for ω_g on $(0, 1)$ is more complicated. Let $x_0 = 0$ and x_n be the n-th smallest local extremum of g on $(0, \infty)$. It is easy to see that $x_{2n-1} = n\pi$. Put $I_n = [x_n, x_{n+1}]$ for $n = 0, 1, 2, \cdots$ and define $g_n = g|_{I_n}$. Finally, put $t_0 = 1$ and t_n be the maximum of g on $[n\pi, (n + 1)\pi]$. If $n \geq 0$ and $t_{n+1} \leq t < t_n$, then we have

$$\omega_g(t) = \sum_{k=0}^{2n} (-1)^k g_k^{-1}(t). \tag{3}$$

The inequality in Lemma 2 yields $\omega_f - \omega_g > 0$ on $(t_1, 1)$. So if we can prove that $\omega_f - \omega_g$ is increasing on $(0, t_1)$, everything is OK. To accomplish this, it suffices to show that $|\omega_g'/\omega_f'| > 1$ on (t_{n+1}, t_n) for positive integer n. Note that from (3), we have

$$|\omega_g'(t)| = -\omega_g'(t) = \sum_{\substack{g(\alpha)=t \\ \alpha>0}} \frac{1}{|g'(\alpha)|}.$$

For $t_{n+1} < t < t_n$, $g(\alpha) = t$ has exactly one solution on each I_k for $k = 0, 1, \cdots, 2n$ and no solution on other I_k. Also, it is easy to show that $|g'(x)| \leq 1/2$ on I_1 and $|g'(x)| \leq 1/(n\pi)$ on $I_{2n-1} \cup I_{2n}$, thus

$$|\omega_g'(t)| \geq 2 + n(n + 1)\pi > \pi(n + \tfrac{3}{2}).$$

Hence

$$|\omega_g'(t)/\omega_f'(t)| > \pi(n + \tfrac{3}{2})t\sqrt{\frac{2}{\pi} \ln\left(\frac{1}{t}\right)}. \tag{4}$$

By simple observation, we have

$$\frac{1}{(n + \frac{1}{2})\pi} \leq t_n \leq \frac{1}{n\pi}.$$

Since $t\sqrt{\log(1/t)}$ is increasing on $(0, 1/\sqrt{e})$ and $(t_{n+1}, t_n) \subset (0, 1/\sqrt{e})$ by the inequality above, we have

$$|\omega_g'(t)/\omega_f'(t)| > \pi(n + \tfrac{3}{2})t_{n+1}\sqrt{\frac{2}{\pi} \ln\left(\frac{1}{t_{n+1}}\right)} \geq \sqrt{\frac{2}{\pi} \ln(2\pi)} > 1,$$

completing the proof. □