Similarity transform

Suppose that we have our favorite matrix aa. It is real and will have distinct real eigen values. We see that from the linear ODE problem, it would be nice to transform this problem into a simpler problem that has only diagonal elements. Can we do this? What values will be on the diagonal of the simpler matrix?

A typical matrix

```
aa = N[{{2, -1, 3}, {2, -4, -3}, {1, 2, 7}}]
```

$$\begin{pmatrix} 2. & -1. & 3. \\ 2. & -4. & -3. \\ 1. & 2. & 7. \end{pmatrix}$$

```
evals = Eigenvalues[aa]
```

{7.2789, −2.44731, 0.16841}

```
evecs = Eigenvectors[aa]
```

$$\begin{pmatrix} 0.508836 & -0.135877 & 0.850072 \\ 0.358301 & 0.904957 & -0.229506 \\ -0.771134 & -0.571679 & 0.280241 \end{pmatrix}$$

We should verify that these are linearly independent and check to see if they happen to be orthogonal.

First, they are linearly independent.

```
Det[evecs]
```

0.470973

Now check the orthogonality

```
evecs⟦2⟧ . evecs⟦2⟧
```

1.

```
evecs[[1]].evecs[[2]]
```

−0.135743

```
evecs[[3]].evecs[[1]]
```

−0.0764779

```
evecs[[2]].evecs[[3]]
```

−0.85796

Now we understand that this transformation can be accomplished using a similarity transform where the form is $P^{-1} A P$. The matrix P is composed of column vectors that are the eigenvectors of A.

```
p = Transpose[evecs]
```

$$\begin{pmatrix} 0.508836 & 0.358301 & -0.771134 \\ -0.135877 & 0.904957 & -0.571679 \\ 0.850072 & -0.229506 & 0.280241 \end{pmatrix}$$

```
pinv = Inverse[p]
```

$$\begin{pmatrix} 0.259892 & 0.162577 & 1.04679 \\ -0.950989 & 1.69461 & 0.840113 \\ -1.56717 & 0.894665 & 1.08108 \end{pmatrix}$$

We can then get back the eigenvalues of aa

```
Chop[pinv . aa . p]
```

$$\begin{pmatrix} 7.2789 & 0 & 0 \\ 0 & -2.44731 & 0 \\ 0 & 0 & 0.16841 \end{pmatrix}$$

■ Now try a symmetric matrix

```
aa = N[{{1, 2, 3}, {2, 0, -3}, {3, -3, 1}}]
```

$$\begin{pmatrix} 1. & 2. & 3. \\ 2. & 0 & -3. \\ 3. & -3. & 1. \end{pmatrix}$$

```
evals = Eigenvalues[aa]
```

$\{-4.69527, 4.22547, 2.4698\}$

```
evecs = Eigenvectors[aa]
```

$$\begin{pmatrix} 0.526837 & -0.605525 & -0.596475 \\ 0.54366 & -0.29937 & 0.784099 \\ -0.653358 & -0.737372 & 0.17148 \end{pmatrix}$$

```
evecs[[1]].evecs[[1]]
```

1.

```
evecs[[1]].evecs[[2]]
```

-4.44089×10^{-16}

```
evecs[[1]].evecs[[3]]
```

9.71445×10^{-17}

```
evecs[[2]].evecs[[3]]
```

-2.77556×10^{-17}

```
p = Transpose[evecs]
```

$$\begin{pmatrix} 0.526837 & 0.54366 & -0.653358 \\ -0.605525 & -0.29937 & -0.737372 \\ -0.596475 & 0.784099 & 0.17148 \end{pmatrix}$$

```
Inverse[p]
```

$$\begin{pmatrix} 0.526837 & -0.605525 & -0.596475 \\ 0.54366 & -0.29937 & 0.784099 \\ -0.653358 & -0.737372 & 0.17148 \end{pmatrix}$$

```
Chop[Inverse[p] . aa . p]
```

$$\begin{pmatrix} -4.69527 & 0 & 0 \\ 0 & 4.22547 & 0 \\ 0 & 0 & 2.4698 \end{pmatrix}$$

```
Chop[Transpose[p] . aa . p]
```

$$\begin{pmatrix} -4.69527 & 0 & 0 \\ 0 & 4.22547 & 0 \\ 0 & 0 & 2.4698 \end{pmatrix}$$

■ The eigenvalues do not have to be real

```
aa = N[{{1, -1, 3}, {2, 0, -3}, {1, 1, 1}}]
```

$$\begin{pmatrix} 1. & -1. & 3. \\ 2. & 0 & -3. \\ 1. & 1. & 1. \end{pmatrix}$$

```
evals = Eigenvalues[aa]
```

$\{2.75531, -0.377654 + 2.22227\,i, -0.377654 - 2.22227\,i\}$

```
evecs = Eigenvectors[aa]
```

$$\begin{pmatrix} 0.853586 & 0.055619 & 0.517975 \\ -0.143161 + 0.448655\,i & 0.825076 + 0.\,i & 0.00842373 - 0.312078\,i \\ -0.143161 - 0.448655\,i & 0.825076 + 0.\,i & 0.00842373 + 0.312078\,i \end{pmatrix}$$

```
p = Transpose[evecs]
```

$$\begin{pmatrix} 0.853586 & -0.143161 + 0.448655\, i & -0.143161 - 0.448655\, i \\ 0.055619 & 0.825076 + 0.\, i & 0.825076 + 0.\, i \\ 0.517975 & 0.00842373 - 0.312078\, i & 0.00842373 + 0.312078\, i \end{pmatrix}$$

```
pinv = Inverse[p]
```

$$\begin{pmatrix} 0.622246 + 4.21067 \times 10^{-18}\, i & 0.0988343 + 0.\, i & 0.894565 - 6.93889 \times 10^{-18}\, i \\ -0.020973 - 0.515825\, i & 0.602673 - 0.0982883\, i & -0.0301516 + 0.860597\, i \\ -0.020973 + 0.515825\, i & 0.602673 + 0.0982883\, i & -0.0301516 - 0.860597\, i \end{pmatrix}$$

```
Chop[pinv . aa . p]
```

$$\begin{pmatrix} 2.75531 & 0 & 0 \\ 0 & -0.377654 + 2.22227\, i & 0 \\ 0 & 0 & -0.377654 - 2.22227\, i \end{pmatrix}$$

Now, what will happen if we do not have distinct eigenvalues.?

Recall our matrix that we used to calculate the generalized eigenvectors

```
aa = {{2, -1, 2, 0}, {0, 3, -1, 0}, {0, 1, 1, 0}, {0, 1, -3, 5}}
```

$$\begin{pmatrix} 2 & -1 & 2 & 0 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -3 & 5 \end{pmatrix}$$

The row matrix of the eigenvectors is

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} \\ 0 & 2 & 1 & \frac{5}{9} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & \frac{2}{3} \\ 0 & 2 & 1 & \frac{5}{9} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

```
p = Transpose[R]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & \frac{2}{3} & \frac{5}{9} & 1 \end{pmatrix}$$

```
pinv = Inverse[p]
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \frac{1}{9} & -\frac{7}{9} & 1 \end{pmatrix}$$

```
simans = pinv . aa . p
```

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$$

We see that the new matrix is much simpler that the original aa, but not exactly diagonal. This is the expected result for a matrix with repeated eigenvalues However, there is a caveat.

Recall this matrix

```
mat={{2,2,1},{1,3,1},{1,2,2}}
```

$$\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

We see that there is a repeated eigenvalue.

```
Eigenvalues[mat]
```

{1, 1, 5}

However, just calculate the eigenvectors and we have:

```
evs=Eigenvectors[mat]
```

$$\begin{pmatrix} -1 & 0 & 1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Thus there was no need to construct a generalized eigen vector. Now try a similarity transform.

```
p=Transpose[%]
```

$$\begin{pmatrix} -1 & -2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

```
Inverse[p].mat.p
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Here is a bonus, the matrix from the Lorenz equations.

```
aa = {{-s, s, 0}, {1, -1, 0}, {0, 0, -b}}
```

$$\begin{pmatrix} -s & s & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

```
MatrixForm[aa]
```

$$\begin{pmatrix} -s & s & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

```
evals = Eigenvalues[aa]
```

$\{0, -b, -s-1\}$

```
evecs = Eigenvectors[aa]
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -s & 1 & 0 \end{pmatrix}$$

```
evecs = {{1, 1, 0}, {-s, 1, 0}, {0, 0, 1}}
```

$$\begin{pmatrix} 1 & 1 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
evecs⟦2⟧ . evecs⟦1⟧
```

$1-s$

```
p = Transpose[evecs]
```

$$\begin{pmatrix} 1 & -s & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
pinv = Inverse[p]
```

$$\begin{pmatrix} \frac{1}{s+1} & \frac{s}{s+1} & 0 \\ -\frac{1}{s+1} & \frac{1}{s+1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can hen get back the eigenvalues of aa

```
pinv . aa . p
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\left(\frac{s}{s+1} + \frac{1}{s+1}\right) s - \frac{s}{s+1} - \frac{1}{s+1} & 0 \\ 0 & 0 & -b \end{pmatrix}$$

```
Simplify[%]
```

```
aa = {{5, 4, 3}, {-1, 0, -3}, {1, -2, 1}}
```